Math 2101

Homework 1

Due: October 7, 2013

Problems marked with $*$ will be marked. The students are advised to work over ALL problems in the assignments, so as to keep pace with the material.

1. (a) Let z be a complex number satisfying $z^{10} + z^9 + \cdots + z + 1 = 0$ Show that

$$w = \frac{z^2 + 1}{z}$$

is a real number.

Since

$$z^n - w^n = (z - w)(z^{n-1} + z^{n-2}w + z^{n-3}w^2 + \cdots + zw^{n-2} + w^{n-1})$$

we have that

$$z^{11} - 1 = (z - 1)(z^{10} + z^9 + \cdots + z + 1) = (z - 1) \cdot 0 = 0.$$

This means that

$$z^{11} = 1 \Longrightarrow |z^{11}| = 1 \Longrightarrow |z|^{11} = 1 \Longrightarrow |z| = 1 \Longrightarrow |z|^2 = z\bar{z} = 1 \Longrightarrow \bar{z} = \frac{1}{z}.$$

Therefore,

$$w = z + \frac{1}{z} = z + \bar{z} = 2\Re(z) \in \mathbb{R}.$$

(b) Find the fourth roots of $16i$.

We write $16i = 16(\cos(\pi/2) + i\sin(\pi/2))$ (polar form) and apply De Moivre's formulas. The fourth roots are:

$$r_k = \sqrt[4]{16}\left(\cos\left(\frac{\pi/2 + 2\pi k}{4}\right) + i\sin\left(\frac{\pi/2 + 2\pi k}{4}\right)\right), \quad k = 0, 1, 2, 3.$$

This gives

$$r_0 = 2(\cos(\pi/8) + i\sin(\pi/8)) = \sqrt{2 + \sqrt{2}} + i\sqrt{2 - \sqrt{2}},$$

$$r_1 = 2(\cos(\pi/8 + \pi/2) + i\sin(\pi/8 + \pi/2)) = 2(-\sin(\pi/8) + i\cos(\pi/8)) = -\sqrt{2 - \sqrt{2}} + i\sqrt{2 + \sqrt{2}}$$

$$r_2 = 2(\cos(\pi/8+\pi) + i\sin(\pi/8+\pi)) = -\sqrt{2+\sqrt{2}} - i\sqrt{2-\sqrt{2}}$$

$$r_3 = 2(\cos(\pi/8+3\pi/2)+i\sin(\pi/8+3\pi/2) = 2(\sin(\pi/8)-i\cos(\pi/8)) = \sqrt{2-\sqrt{2}}-i\sqrt{2+\sqrt{2}}.$$

The calculation of the trigonometric numbers follows from the identity

$$\cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x,$$

which gives

$$\cos(\pi/4) = 2\cos^2(\pi/8)-1 = 1-2\sin^2(\pi/8) \Longrightarrow \cos(\pi/8) = \sqrt{\frac{2+\sqrt{2}}{4}}, \sin(\pi/8) = \sqrt{\frac{2-\sqrt{2}}{4}}.$$

We also used other well-known trig formulas e.g. $\sin(x+\pi/2) = \cos x$, $\cos(x+\pi/2) = -\sin x$.

2. $*$ Let a and b be real numbers, z a complex number such that

$$a \cdot i^{-2012} + b \cdot i^{2010} = |2z+2| i^{2011} - |z-1| i^{-2013}.$$

Where are the images of z in the complex plane located?

We recall the powers of i:

$$i^n = \begin{cases} i, & n \equiv 1 \bmod 4 \\ -1, & n \equiv 2 \bmod 4 \\ -i, & n \equiv 3 \bmod 4 \\ 1, & n \equiv 0 \bmod 4 \end{cases}$$

So we have

$$i^{-2012} = 1, \quad i^{2010} = -1, \quad i^{2011} = -i, \quad i^{-2013} = -i.$$

Therefore,

$$a \cdot i^{-2012} + b \cdot i^{2010} = a - b \in \mathbb{R}, \quad |2z+2| i^{2011} - |z-1| i^{-2013} = -i|2z+2| + i|z-1| = i(|z-1| - |2z+2|).$$

The only real number which is simultaneously purely imaginary is 0, so

$$|z-1| - |2z+2| = 0.$$

We rewrite it to find the locus of such z:

$$|z-1| = |2z+2| \Leftrightarrow |z-1|^2 = |2z+2|^2 \Leftrightarrow (z-1)(\bar{z}-1) = (2z+2)(2\bar{z}+2)$$

$$\Leftrightarrow z\bar{z} - z - \bar{z} + 1 = 4z\bar{z} + 4z + 4\bar{z} + 4 \Leftrightarrow 3z\bar{z} + 5z + 5\bar{z} + 3 = 0 \Leftrightarrow 3(x^2+y^2) + 10x + 3 = 0$$

$$\Leftrightarrow x^2 + y^2 + \frac{10}{3}x + 1 = 0 \Leftrightarrow (x+\frac{5}{3})^2 + y^2 - \frac{25}{9} + 1 = 0 \Leftrightarrow (x+\frac{5}{3})^2 + y^2 = \frac{16}{9}.$$

This is the equation of a circle of radius $4/3$ centered at $(-5/3, 0)$.

3. Given the equation $z^2 - az + b = 0$ with $a, b \in \mathbb{R}$ and z_1 and z_2 its roots. We assume that $z_1 = 2 + i$.

   (a) Find a and b.
   Since a and b are real, the roots of the equation are conjugate pairs, i.e. if z_1 is a root, then $\bar{z}_1$ is a root. Therefore, $z_2 = 2 - i$.
   The equation with these roots and leading coefficient 1 is

   $$(z-z_1)(z-z_2) = 0 \Leftrightarrow (z-2-i)(z-2+i) = 0 \Leftrightarrow (z-2)^2-i^2 = 0 \Leftrightarrow z^2-4z+4+1 = 0$$

   This gives $a = 4$ and $b = 5$.

   (b) Show that $z_1^{2013} + z_2^{2013}$ is real and write an expression for it in terms of trigonometric numbers showing that it is real.
   The easy way to see this is as follows:

   $$z_1^{2013} + z_2^{2013} = z_1^{2013} + (\bar{z}_1)^{2013} = z_1^{2013} + \overline{(z_1^{2013})} = 2\Re(z_1^{2013}),$$

   using the property that $(\bar{w})^n = \overline{w^n}$.
   We can use De Moivre's formula for this as follows: Let $t = \arg(2 + i)$ so that

   $$z_1 = 2 + i = |2 + i|(\cos t + i \sin t) = \sqrt{5}(\cos t + i \sin t)$$

   in polar form. At the same time the polar form of $2 - i$ is

   $$z_2 = 2 - i = |2 - i|(\cos(-t) + i \sin(-t) = \sqrt{5}(\cos(-t) + i \sin(-t)).$$

   We recall here that the arguments of complex conjugates are opposite. Using De Moivre's formula we get

   $$z_1^{2013} = (\sqrt{5})^{2013}(\cos(2013t) + i \sin(2013t)),$$

   $$z_2^{2013} = (\sqrt{5})^{2013}(\cos(-2013t)+i \sin(-2013t)) = (\sqrt{5})^{2013}(\cos(2013t)-i \sin(2013t)).$$

   Therefore,

   $$z_1^{2013} + z_2^{2013} = 2(\sqrt{5})^{2013} \cos(2013t) \in \mathbb{R}.$$

   (c) If A, B, C correspond to z_1, z_2 and z_3 in the plane with

   $$z_3 = \frac{z_1}{z_2} + \frac{1}{5}(17 + i),$$

   show that the triangle ABC is a right-isosceles triangle.
   We calculate z_3:

   $$z_3 = \frac{z_1}{z_2}+\frac{1}{5}(17+i) = \frac{2 + i}{2 - i}+\frac{1}{5}(17+i) = \frac{(2 + i)^2}{|2 - i|^2}+\frac{17 + i}{5} = \frac{3 + 4i}{5}+\frac{17 + i}{5} = \frac{20 + 5i}{5} = 4+i.$$

We see that the segment AC is horizontal, as z_1, z_3 have the same imaginary part. Moreover, the segment AB is vertical, as z_1, z_2 are complex conjugates. Therefore the triangle has a right angle at A. A different proof can be based on the fact that perpendicular lines have slopes with product -1, which translates into the quotient of the complex numbers being purely imaginary. We have

$$z_3 - z_1 = -i(z_1 - z_2),$$

which is true as $z_3 - z_1 = 4 + i - 2 - i = 2$, while $z_1 - z_2 = 2 + 1 - (2 - i) = 2i$. The triangle is also isosceles, as $AB = AC$ i.e

$$|z_3 - z_1| = |z_2 - z_1| \Leftrightarrow |2| = |2i|.$$

(d) If $|w - z_1| = |\bar{w} - z_1|$ show that $w \in \mathbb{R}$.
There are two methods again, the geometric and the analytic. Make yourself familiar with both!
(a) Since $|\bar{w} - z_1| = |\bar{w} - \bar{z}_2| = |\overline{w - z_2}| = |w - z_2|$ we have that $|w - z_1| = |w - z_2|$, i.e. the point representing w is equidistant from A and B. These points are in the midpoint bisector of the segment AB, i.e. the real line, as z_1 and z_2 are complex conjugate.
(b) We have

$$|w - z_1| = |\bar{w} - z_1| \Leftrightarrow |w - z_1|^2 = |\bar{w} - z_1|^2 \Leftrightarrow (w - z_1)(\bar{w} - \bar{z}_1) = (\bar{w} - z_1)(w = \bar{z}_1)$$

$$\Leftrightarrow w\bar{w} - z_1\bar{w} - \bar{z}_1 w + z_1\bar{z}_1 = \bar{w}w - z_1 w - \bar{z}_1\bar{w} + z_1\bar{z}_1 \Leftrightarrow \bar{w}(\bar{z}_1 - z_1) = w(\bar{z}_1 - z_1) \Leftrightarrow \bar{w} = w,$$

as $z_1 \neq \bar{z}_1$. Since $w = \bar{w}$, we have $w \in \mathbb{R}$.

4. $*$ Let $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$ with $a \neq b$ and $n \in N$ such that

$$(1 + iz)^n = \frac{a + bi}{b + ai}.$$

(a) Show that z is not a real number.
We have that

$$\left|\frac{a + bi}{b + ai}\right| = \frac{|a + bi|}{|b + ai|} = \frac{\sqrt{a^2 + b^2}}{\sqrt{b^2 + a^2}} = 1.$$

On the other hand, if $z \in \mathbb{R}$, and $z \neq 0$

$$|(1 + iz)^n| = |1 + iz|^n = (\sqrt{1 + z^2})^n > 1^n = 1.$$

This is a contradiction. If $z = 0$, then

$$1 = \frac{a + bi}{b + ai} \Longrightarrow a + bi = b + ai \Longrightarrow a = b$$

as $a, b \in \mathbb{R}$. This is a contradiction to $a \neq b$. So $z \notin \mathbb{R}$.

(b) Show that z lies on a circle. Find the radius and centre of the circle.
By the argument above, we have that

$$|1+iz|^n = 1 \Longrightarrow |1+iz| = 1 \Longrightarrow |i(-i+z)| = 1 \Longrightarrow |z-i| = 1.$$

This is a circle of radius 1 centered at i.

(c) Find the maximum modulus of such z.
The point further away from 0 on this circle is $2i$. It has modulus 2.

(d) Show that $4 < |z-3+4i| < 7$.
We use the triangle inequality $||z|-|w|| \leq |z-w| \leq |z|+|w|$. We have

$$||i-3+4i|-|z-i|| \leq |z-3+4i| \leq |z-i|+|i-3+4i| = 1+|-3+5i| = 1+\sqrt{9+25} = 1+\sqrt{34} < 7,$$

as $\sqrt{34} < \sqrt{36} = 6$. The left-hand side of the inequality is $\sqrt{34}-1 > 4 \Leftrightarrow \sqrt{34} > 5 = \sqrt{25}$.

5. Prove the Lagrange identity

$$\left|\sum_{i=1}^n a_i b_i\right|^2 = \sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2 - \sum_{1\leq i<j\leq n} |a_i\bar{b}_j - a_j\bar{b}_i|^2.$$

We need to show the equivalent formula

$$\left|\sum_{i=1}^n a_i b_i\right|^2 + \sum_{1\leq i<j\leq n} |a_i\bar{b}_j - a_j\bar{b}_i|^2 = \sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2.$$

We expand out using $|z|^2 = z\bar{z}$ to get

$$\left(\sum_{i=1}^n a_i b_i\right)\left(\sum_{j=1}^n \bar{a}_j\bar{b}_j\right) + \sum_{1\leq i<j\leq n} (a_i\bar{b}_j - a_j\bar{b}_i)(\bar{a}_i b_j - \bar{a}_j b_i)$$

$$= \sum_{i,j=1}^n a_i b_i \bar{a}_j \bar{b}_j + \sum_{i<j}(|a_i|^2|b_j|^2 + |a_j|^2|b_i|^2 - a_i\bar{a}_j b_i\bar{b}_j - a_j\bar{a}_i b_j\bar{b}_i)$$

$$= \sum_{i\neq j} a_i b_i \bar{a}_j \bar{b}_j + \sum_{i=j=1}^n |a_i|^2|b_i|^2 + \sum_{i<j}(|a_i|^2|b_j|^2 + |a_j|^2|b_i|^2) - \sum_{i<j}(a_i\bar{a}_j b_i\bar{b}_j + a_j\bar{a}_i b_j\bar{b}_i).$$

The first and the last sum are equal and, therefore, cancel. This can be seen as follows: In the last sum we have the condition $i<j$ while in the first only $i \neq j$. A pair (i,j) of unequal integers has either $i<j$ or $j<i$. In the second case $a_i b_i \bar{a}_j \bar{b}_j = a_{j'} b_{j'} \bar{a}_{i'} \bar{b}_{i'}$ with $j = i' < j' = i$, so we get the term $a_j b_j \bar{a}_i \bar{b}_i$ with $i<j$. The two summands in the middle give exactly

$$\sum_{i=1}^n |a_i|^2 \sum_{j=1}^n |b_j|^2$$

by the distributive law and the same thinking about $i \neq j$ vs $i<j$ and $j<i$.

6. Prove that
$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1,$$
if $|a| < 1$ and $|b| < 1$.

We have
$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1 \Leftrightarrow |a-b| < |1-\bar{a}b| \Leftrightarrow |a-b|^2 < |1-\bar{a}b|^2 \Leftrightarrow |a|^2+|b|^2-2\Re(a\bar{b}) < 1+|\bar{a}b|^2-2\Re(\bar{a}b)$$
(here we notice that $\Re(\bar{a}b) = \Re(a\bar{b})$ as they are conjugate numbers)
$$\Leftrightarrow |a|^2 + |b|^2 < 1 + |a|^2|b|^2 \Leftrightarrow 0 < 1 + |a|^2|b|^2 - |a|^2 - |b|^2 = (1-|a|^2)(1-|b|^2).$$
The assumption $|a| < 1$ and $|b| < 1$ gives the result.

7. $*$ Let z_1, z_2, z_3 be three complex numbers with $|z_j| = 2$ for $j = 1, 2, 3$. We are given that
$$\Re\left(\frac{z_1}{z_2}\right) = \Re\left(\frac{z_2}{z_3}\right) = \Re\left(\frac{z_3}{z_1}\right) = -\frac{1}{2}.$$
Show that $z_1 + z_2 + z_3 = 0$ and that the triangle with vertices z_1, z_2, z_3 is equilateral.

Hint: What happens when we rotate around 0 an equilateral triangle centered at 0? Why can we assume that z_1 is real and positive?

Fix θ. Let us write $z_k = w_k e^{i\theta}$ for $k = 1, 2, 3$. Then for the quotients z_k/z_j we have
$$\frac{z_k}{z_j} = \frac{w_k}{w_j}.$$
This means that the relations
$$\Re\left(\frac{z_1}{z_2}\right) = \Re\left(\frac{z_2}{z_3}\right) = \Re\left(\frac{z_3}{z_1}\right) = -\frac{1}{2}$$
are equivalent to
$$\Re\left(\frac{w_1}{w_2}\right) = \Re\left(\frac{w_2}{w_3}\right) = \Re\left(\frac{w_3}{w_1}\right) = -\frac{1}{2}.$$
If we assume that w_1 is real and positive, i.e. $\theta = \arg z_1$, and we prove that
$$w_1 + w_2 + w_3 = 0$$
and the triangles with vertices corresponding to w_1, w_2, w_3 is equilateral, then by multiplying with $e^{i\theta}$ we get
$$z_1 + z_2 + z_3 = 0.$$
Equilateral triangles remain equilateral triangles when rotated around 0, so the triangle with vertices z_1, z_2, z_3 is equilateral. So we can assume that z_1 is real and positive.

Since $|z_1| = 2$, we assume here that $z_1 = 2$. Then

$$-\frac{1}{2} = \Re(z_1/z_2) = z_1\Re(1/z_2) = 2\frac{\Re(z_2)}{|z_2|^2} = \frac{2\Re(z_2)}{4} \Longrightarrow \Re(z_2) = -1.$$

We have used that $|z_2| = 2$. Using $\Re(z_3/z_1) = -1/2$ and $z_1 = 2$ we get that $\Re(z_3) = -1$. We can conclude that

$$\Re(z_1 + z_2 + z_3) = \Re(z_1) + \Re(z_2) + \Re(z_3) = 2 - 1 - 1 = 0.$$

To show that $z_1 + z_2 + z_3 = 0$ it suffices to show that $\Im(z_1 + z_2 + z_3) = 0$, i.e. $\Im(z_2 + z_3) = 0$, as $\Im(z_1) = 0$. Since $|z_2| = |z_3| = 2$ we get from $\Re(z_2) = \Re(z_3) = -1$ that $|\Im(z_2)| = |\Im(z_3)| = \sqrt{3}$. This means that $\Im(z_2) = \pm\sqrt{3}$ and $\Im(z_3) = \pm\sqrt{3}$. We need to show that the signs have to be taken to be opposite. It is here that we use

$$\Re\left(\frac{z_2}{z_3}\right) = -1/2 \Leftrightarrow \Re\left(\frac{z_2\bar{z}_3}{|z_3|^2}\right) = -1/2 \Leftrightarrow \Re(z_2\bar{z}_3) = -2 \Leftrightarrow \Re(z_2)\Re(z_3)+\Im(z_2)\Im(z_3) = -2$$

$$\Leftrightarrow 1 + \Im(z_2)\Im(z_3) = -2 \Leftrightarrow \Im(z_2)\Im(z_3) = -3.$$

We have arrived at $z_2 = -1 \pm i\sqrt{3}, z_3 = -1 \mp i\sqrt{3}$, where the signs are opposite. We now calculate

$$\begin{aligned}|z_1 - z_2| &= |2 + 1 \mp i\sqrt{3}| = \sqrt{9+3} = \sqrt{12},\\ |z_1 - z_3| &= |2 + 1 \pm i\sqrt{3}| = \sqrt{9+3} = \sqrt{12},\\ |z_2 - z_3| &= |-1 \pm i\sqrt{3} + 1 \pm i\sqrt{3}| = |2i\sqrt{3}| = \sqrt{12}.\end{aligned}$$

So the triangle is equilateral.

8. Express $\cos(5\phi)$ in terms of $\cos(\phi)$. Express $\sin(5\phi)$ in terms of $\sin(\phi)$.

   By de Moivre's formula

$$(\cos\phi + i\sin\phi)^n = \cos(n\phi) + i\sin(n\phi)$$

for $n = 5$ we get by the binomial theorem

$$(\cos\phi+i\sin\phi)^5 = \cos^5\phi+i5\cos^4\sin\phi-10\cos^3\phi\sin^2\phi-i10\cos^2\phi\sin^3\phi+5\cos\phi\sin^4\phi+i\sin^5\phi.$$

$$\cos(5\phi) = \Re((\cos\phi + i\sin\phi)^5) = \cos^5\phi - 10\cos^3\phi\sin^2\phi + 5\cos\phi\sin^4\phi$$

$$= \cos^5\phi - 10\cos^3\phi(1-\cos^2\phi) + 5\cos\phi(1-\cos^2\phi)^2 = 16\cos^5\phi - 20\cos^3\phi + 5\cos\phi$$

$$\sin(5\phi) = \Im((\cos\phi + i\sin\phi)^5) = 5\cos^4\phi\sin\phi - 10\cos^2\phi\sin^3\phi + \sin^5\phi$$

$$= 5\sin\phi(1-\sin^2\phi)^2 - 10(1-\sin^2\phi)\sin^3\phi + \sin^5\phi = 16\sin^5\phi - 20\sin^3\phi + 5\sin\phi.$$

9. In a triangle a median is the line joining a vertex with the midpoint of the opposite side. Show that the three medians intersect at the same point. This point is called the barycenter. If z_i are complex numbers corresponding to the vertices of the triangle, which complex number corresponds to the barycenter?

   You must use complex numbers, not geometric arguments.

   If A, B, C are the vertices of the triangle, corresponding to z_1, z_2, z_3 we will show that the point M (barycenter) corresponding to $(z_1 + z_2 + z_3)/3$ is the common point of the three medians. Moreover, the barycenter splits the median in proportion $2 : 1$.

   We will use the following fact: If $D(x_1, y_1)$ and $E(x_2, y_2)$ are two points in the plane corresponding to the complex numbers $x_j + iy_j$ respectively, $i = 1, 2$, then the midpoint on the segment DE is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \frac{z_1 + z_2}{2}.$$

   We get that the midpoint A_1 of BC corresponds to $(z_2 + z_3)/2$, the midpoints B_1 of AC is $(z_1 + z_3)/2$, and the midpoint C_1 of AB is $(z_1 + z_2)/2$. The vector $A\vec{A}_1$ corresponds to the complex number

$$\frac{z_2 + z_3}{2} - z_1 = \frac{-2z_1 + z_2 + z_3}{2}.$$

   Since M should split the median in ratio $2 : 1$ we calculate that $2/3$ of vector $A\vec{A}_1$ corresponds to the complex number

$$\frac{2}{3}\left(\frac{z_2 + z_3}{2} - z_1\right) = \frac{-2z_1 + z_2 + z_3}{3}.$$

   Let M_1 be the corresponding point on the median, so that $AM_1 = 2/3AA_1$ (we will see that it is the point M). Then it corresponds to the complex number

$$z_1 + \frac{-2z_1 + z_2 + z_3}{3} = \frac{3z_1 - 2z_1 + z_2 + z_3}{3} = \frac{z_1 + z_2 + z_3}{3},$$

   which is the point M. Here we added z_1 as we perform the addition of the vectors $O\vec{A}$ and AM_1. We can now repeat for the other two medians and find that the point on them with ratio $2 : 1$ is always the point M. So M is the common point of all three medians.

10. $*$ Prove that each of the following sets are open:

    (a) $A = \{z : \Im(z) < 0\}$,

    Given $z \in A$, we need to find an open neighborhood (ball) $D(z, r)$, $r > 0$ such that $D(z, r) \subset A$. The picture below will convince you that we can choose $r = |\Im(z)|$, which is > 0, by the assumption $z \in A$. So we need to prove that $D(z, |\Im(z)|) \subset A$. Let $w \in D(z, |\Im(z)|)$. By the definition of $D(z, r)$, we have that

$$|w - z| < |\Im(z)|.$$

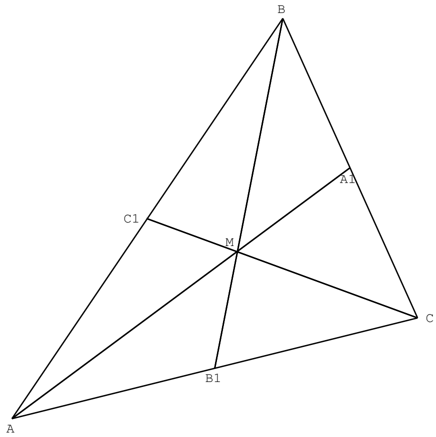


Figure 1: The medians of the triangle

Since the modulus is larger (or equal) to the imaginary part of a complex number: $|\Im(x+iy)| \leq |x+iy|$, we get (using transitivity)

$$|\Im(w-z)| < |\Im(z)|.$$

The triangle inequality for real numbers gives:

$$\Im(w) - \Im(z) = \Im(w-z) \leq |\Im(w-z)| < |\Im(z)| = -\Im(z) \Longrightarrow \Im(w) - \Im(z) < -\Im(z).$$

This implies that

$$\Im(w) < 0 \Longrightarrow w \in A,$$

as required.

(b) $B = \{z : \Re(z) > 0 \text{ and } \Im(z) > 0\}$.

Given $z \in B$, we need to find an open neighborhood (ball) $D(z,r)$, $r > 0$ such that $D(z,r) \subset B$. The picture will convince you that we can choose $r = \min(\Im(z), \Re(z))$. This is positive as the minimum of two positive numbers, as $\Re(z) > 0$ and $\Im(z) > 0$, by the assumption $z \in B$. So we need to prove that $D(z,r) \subset B$. Let $w \in D(z,r)$. By the definition of $D(z,r)$, we have that

$$|w-z| < r = \min(\Im(z), \Re(z)).$$

Since the modulus is larger (or equal) to the real part of a complex number: $|\Re(x+iy)| \leq |x+iy|$, we get (using transitivity)

$$|\Re(w-z)| < \Re(z).$$

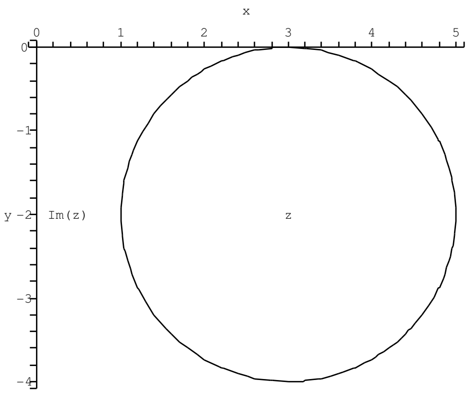
The triangle inequality for real numbers gives:

$$\Re(z) - \Re(w) = \Re(z-w) \leq |\Re(z-w)| = |\Re(w-z)| < \Re(z) \Longrightarrow \Re(z) - \Re(w) < \Re(z).$$

This implies that $\Re(w) > 0$. We need to show the second condition as well: $\Im(w) > 0$. We get similarly to (a) $|\Im(w-z)| < \Im(z)$. This implies

$$\Im(z) - \Im(w) = \Im(z-w) \leq |\Im(w-z)| < \Im(z) \Longrightarrow \Im(z) - \Im(w) < \Im(z).$$

This gives $\Im(w) > 0$ as required.



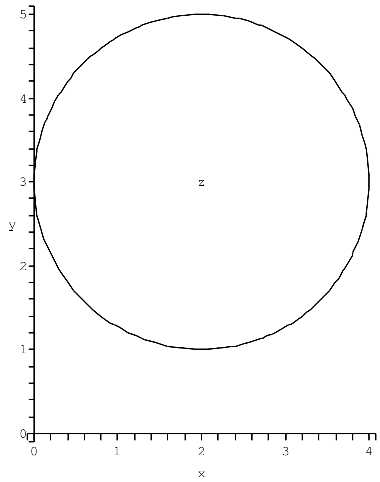


Figure 2: The pictures for (a) and (b) respectively

Math 2101

Homework 2

Due: October 14, 2013

1. $*$ Verify the Cauchy-Riemann equations for

$$\text{(a) } f(z) = e^{-x}(\cos y - i \sin y), \quad \text{(b) } f(z) = \cos x \cosh y - i \sin x \sinh y.$$

Find $f'(z)$ in both cases.

For the function $f(z) = e^{-x}(\cos y - i\sin y)$ we have $u = e^{-x}\cos y$ and $v = -e^{-x}\sin y$. Therefore,

$$\frac{\partial u}{\partial x} = -e^{-x}\cos y = \frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial y} = -e^{-x}\sin y, \quad \text{while} \quad \frac{\partial v}{\partial x} = e^{-x}\sin y = -\frac{\partial u}{\partial y}.$$

Since $f'(z) = u_x + iv_x$, we get that $f'(z) = -e^{-x}\cos y + ie^{-x}\sin y$.

For the function $f(z) = \cos x \cosh y - i \sin x \sinh y$ we have $u = \cos x \cosh y$ and $v = -\sin x \sinh y$. Therefore,

$$\frac{\partial u}{\partial x} = -\sin x \cosh y = \frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial y} = \cos x \sinh y, \quad \text{while} \quad \frac{\partial v}{\partial x} = -\cos x \sinh y = -\frac{\partial u}{\partial y}.$$

Similarly, $f'(z) = -\sin x \cosh y - i \cos x \sinh y$.

2. $*$ Apply the definition of derivatives to show that if $f(z) = \Re(z)$, then $f'(z)$ does not exist anywhere.

If $f(z)$ is differentiable at a point $z_0 = x_0 + iy_0$, then the following limit exists:

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z\to z_0} \frac{\Re(z) - \Re(z_0)}{z - z_0} = \lim_{z\to z_0} \frac{x - x_0}{z - z_0}.$$

We will show that this limit does not exist by letting z approach z_0 in two ways:

(a) z approaches z_0 horizontally, i.e. $z - z_0$ is real, i.e. $z - z_0 = \Re(z - z_0)$. Then the quotient in the limit above is

$$\frac{x - x_0}{z - z_0} = \frac{x - x_0}{\Re(z - z_0)} = \frac{x - x_0}{x - x_0} = 1,$$

and the limit is 1.

(b) z approaches z_0 vertically, i.e. $z - z_0$ is purely imaginary, i.e. $z - z_0 = i(y - y_0)$ and $x = x_0$. Then the quotient in the limit above is

$$\frac{x - x_0}{z - z_0} = \frac{x - x_0}{z - z_0} = \frac{x_0 - x_0}{z - z_0} = 0,$$

and the limit is 0.

3. Suppose that f is holomorphic in a region Ω. Prove that in any of the following cases

   (a) $\Re(f)$ is constant;

   (b) $\Im(f)$ is constant;

   (c) f is real-valued;

   (d) $\arg(f)$ is constant;

   we can conclude that f is a constant. Do not use integration.

   Hint for (d): Can you find a complex number c such that $g(z) = cf(z)$ is real-valued?

   For (a) we note that if u is constant, then $u_x = 0$ and $u_y = 0$. But $f'(z) = u_x + iv_x = u_x - iu_y = 0$. So Theorem 11(a) gives that f is constant. For (b) we note that if v is constant, then $v_x = v_y = 0$. But $f'(z) = v_y + iv_x = 0$. So again f is constant. For (c) we notice that f real-valued, means $v = 0$, i.e. v is constant. Now use (b). For (d) we notice that if the argument of f is constant, the slope of $f(z)$ is constant and this means that we can find a k with $u = kv$ for fixed $k \in \mathbb{R}$. Consider the analytic function $g(z) = (k - i)(u + iv)$, which has imaginary part $kv - u = 0$. Then by (c), $g(z)$ is constant, which implies that f is constant.

4. $*$ (a) Let

$$u(x, y) = \sinh(x) \cdot \sin(y).$$

   Show that u is a harmonic function. Find all functions $v(x, y)$ such that

$$f(x + iy) = u(x, y) + iv(x, y)$$

   is holomorphic. Write f as a function of z.

   Hint: Recall that

$$\sinh(z) = \frac{e^z - e^{-z}}{2}, \quad \cosh(z) = \frac{e^z + e^{-z}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

   A harmonic function u satisfies the Laplace Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

In our case we have

$$u_x = \cosh(x)\sin(y), u_{xx} = \sinh(x)\sin(y), u_y = \sinh(x)\cos(y), u_{yy} = \sinh(x)\cdot(-\sin(y)).$$

Therefore,

$$u_{xx} + u_{yy} = \sinh(x)\sin(y) + \sinh(x)\cdot(-\sin(y)) = 0.$$

We find all harmonic conjugates $v(x, y)$. By the Cauchy–Riemann equations we get:

$$v_y = u_x = \cosh(x)\sin(y) \Longrightarrow v(x,y) = \int \cosh(x)\sin(y)\,dy = -\cosh(x)\cos(y) + c(x)$$

for a constant $c(x)$ that may depend on x but not on y. We now differentiate and compare with the second Cauchy–Riemann equation

$$v_x = -\sinh(x)\cos(y) + c'(x) = -u_y = -\sinh(x)\cos(y).$$

Therefore, $c'(x) = 0 \Longrightarrow c(x) = c$, a constant that does not depend on x and y. We get

$$v(x,y) = -\cosh(x)\cos(y) + c.$$

Now

$$f(z) = \sinh(x)\cdot\sin(y) + i(-\cosh(x)\cos(y) + c) = \frac{e^x - e^{-x}}{2}\frac{e^{iy} - e^{-iy}}{2i} - i\frac{e^x + e^{-x}}{2}\frac{e^{iy} + e^{-iy}}{2} + ic$$

$$-\frac{i}{4}\left(e^{x+iy} - e^{x-iy} - e^{-x+iy} + e^{-x-iy} + e^{x+iy} + e^{x-iy} + e^{-x+iy} + e^{-x-iy}\right) + ic$$

$$= -\frac{i}{4}(2e^{x+iy} + 2e^{-x-iy}) + ic = -\frac{i}{2}(e^z + e^{-z}) = -i\cosh(z) + ic.$$

(b) For the harmonic function $v : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ given by the formula

$$v(x,y) = \frac{-y}{x^2 + y^2}$$

find all holomorphic functions $f(z)$ such that $\Im f(z) = v(x, y)$. Write f as a function of z.

We use the second of the Cauchy–Riemann equations. We have

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{-y}{x^2 + y^2}\right) = -\frac{2xy}{(x^2 + y^2)^2}.$$

For the integration below we substitute $x^2 + y^2 = q$ so that $2y\,dy = dq$ and

$$u(x,y) = \int -\frac{2xy}{(x^2 + y^2)^2}\,dy = \int \frac{-x\,dq}{q^2} = \frac{x}{q} = \frac{x}{x^2 + y^2} + c(x),$$

where $c(x)$ is the constant of integration, depending possibly on x but not on y. We need to find $c(x)$. We differentiate with respect to x and use the first Cauchy–Riemann equation.

$$\frac{\partial u}{\partial x} = \frac{1(x^2+y^2) - 2x \cdot x}{(x^2+y^2)^2} + c'(x) = \frac{y^2 - x^2}{(x^2+y^2)^2} + c'(x) = \frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

the last by direct calculation. We deduce that $c'(x) = 0 \implies c(x) = k$ a (real) constant independent of both x and y. So

$$u(x,y) = \frac{x}{x^2+y^2} + k$$

$$f(z) = u(x,y) + iv(x,y) = \frac{x}{x^2+y^2} + k + i\frac{-y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} + k = \frac{\bar{z}}{|z|^2} + k = \frac{1}{z} + k.$$

Alternative solution: We notice that $g(z) = 1/z$ is holomorphic and has $\Im g(z) = \frac{-y}{x^2+y^2}$. For any solution $f(z)$ we will have $\Im(f(z) - g(z)) = 0$, i.e. the holomorphic function $f - g$ has constant imaginary part. Therefore, by a Problem 3(b), it is a constant on the region, i.e.

$$f(z) = g(z) + k \quad \forall z \in \mathbb{C} \setminus \{0\}.$$

This constant k is real as $\Im(f(z) - g(z)) = 0$.

5. Prove rigorously that, if the function $f(z)$ is holomorphic on $D(0,R)$, then $g(z) = \overline{f(\bar{z})}$ is also holomorphic on $D(0,R)$.

We note that $z \to z_0 \Leftrightarrow \bar{z} \to \bar{z}_0$, as these are both equivalent to $x \to x_0$ and $y \to y_0$. We use the definition of the derivative of g at $z_0 \in D(0,R)$. We notice that this disc is symmetric with respect to the real axis, i.e. $z \in D(0,R) \Leftrightarrow \bar{\in} D(0,R)$. We have

$$\lim_{z\to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z\to z_0} \frac{\overline{f(\bar{z})} - \overline{f(\bar{z}_0)}}{z - z_0} = \lim_{z\to z_0} \overline{\frac{f(\bar{z}) - f(\bar{z}_0)}{\bar{z} - \bar{z}_0}} = \overline{\lim_{z\to z_0} \frac{f(\bar{z}) - f(\bar{z}_0)}{\bar{z} - \bar{z}_0}}$$
$$= \overline{\lim_{\bar{z}\to \bar{z}_0} \frac{f(\bar{z}) - f(\bar{z}_0)}{\bar{z} - \bar{z}_0}} = \overline{\lim_{w\to \bar{z}_0} \frac{f(w) - f(\bar{z}_0)}{w - \bar{z}_0}} = \overline{f'(\bar{z}_0)},$$

with the substitution $w = \bar{z}$ and using the definition of the derivative of f at $\bar{z}_0$.

Remark: It is not enough to check the Cauchy–Riemann equations, as these are not equivalent to being holomorphic. In fact, the calculation below, showing that g satisfies the Cauchy–Riemann equations iff f does is not even easier and uses the chain rule from analysis.

If $f(z) = u(x,y) + iv(x,y)$, then $\overline{f(\bar{z})} = u(x,-y) - iv(x,-y)$. We calculate the partial derivatives with respect to x and y for $\overline{f(\bar{z})}$ using the chain rule from multivariable calculus:

$$\frac{\partial u(x,-y)}{\partial x} = \frac{\partial u}{\partial x}(x,-y)\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}(x,-y)\frac{\partial(-y)}{\partial x} = \frac{\partial u}{\partial x}(x,-y).$$

$$\frac{\partial u(x,-y)}{\partial y} = \frac{\partial u}{\partial x}(x,-y)\frac{\partial x}{\partial y} + \frac{\partial u}{\partial y}(x,-y)\frac{\partial(-y)}{\partial y} = -\frac{\partial u}{\partial y}(x,-y).$$

$$\frac{\partial(-v(x,-y))}{\partial x} = \frac{\partial(-v)}{\partial x}(x,-y)\frac{\partial x}{\partial x} + \frac{\partial(-v)}{\partial y}(x,-y)\frac{\partial(-y)}{\partial x} = -\frac{\partial v}{\partial x}(x,-y).$$

$$\frac{\partial(-v(x,-y))}{\partial y} = \frac{\partial(-v)}{\partial x}(x,-y)\frac{\partial x}{\partial y} + \frac{\partial(-v)}{\partial y}(x,-y)\frac{\partial(-y)}{\partial y} = -\frac{\partial(-v)}{\partial y}(x,-y) = \frac{\partial v}{\partial y}(x,-y).$$

This gives:

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} \Leftrightarrow \frac{\partial u(x,-y)}{\partial x} = \frac{\partial(-v(x,-y))}{\partial y}$$

and

$$\frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x} \Leftrightarrow \frac{\partial u(x,-y)}{\partial y} = -\frac{\partial(-v(x,-y))}{\partial x}.$$

This proves the equivalence of the Cauchy–Riemann equations.

6. Recall the operators $\partial/\partial z$ and $\partial/\partial \bar{z}$ defined by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$

Suppose that U and V are open sets in the complex plane. Assume that $f : U \to V$ and $g : V \to \mathbb{C}$ are two functions that are differentiable in the real sense i.e. $w = f(x,y)$ has continuous partial derivatives in x and y, and $g(u,v)$ has continuous partial derivatives in u and v. We set $h = g \circ f$. Write $\partial g/\partial w$ and $\partial g/\partial \bar{w}$ as functions of $\partial g/\partial u$ and $\partial g/\partial v$. Show that the complex version of the chain rule is

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial w}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{w}}\frac{\partial \bar{f}}{\partial z}, \quad \frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial w}\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{w}}\frac{\partial \bar{f}}{\partial \bar{z}}.$$

This is an exercise in the chain rule from Methods 2.

Set $f(x,y) = u(x,y) + iv(x,y)$, i.e., $(x,y) \to (u(x,y), v(x,y)) \to h(u,v)$.

By the definition of $\partial/\partial z$ and $\partial/\partial \bar{z}$ but applied to $g(u,v)$ we have

$$\begin{aligned}\frac{\partial g}{\partial w} &= \frac{1}{2}\left(\frac{\partial g}{\partial u} - i\frac{\partial g}{\partial v}\right)\\ \frac{\partial g}{\partial \bar{w}} &= \frac{1}{2}\left(\frac{\partial g}{\partial u} + i\frac{\partial g}{\partial v}\right).\end{aligned}$$

We add and subtract the last two equations to get

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial w} + \frac{\partial g}{\partial \bar{w}}, \quad \frac{\partial g}{\partial v} = \frac{1}{i}\left(\frac{\partial g}{\partial \bar{w}} - \frac{\partial g}{\partial w}\right). \tag{1}$$

By the standard chain rule for functions of two variables we have

$$\begin{aligned}\frac{\partial h}{\partial x} &= \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x} \\ \frac{\partial h}{\partial y} &= \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y}.\end{aligned} \tag{2}$$

Moreover,

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}(u_x + iv_x - i(u_y + iv_y)) \\ \frac{\partial \bar{f}}{\partial z} &= \frac{1}{2}\left(\frac{\partial \bar{f}}{\partial x} - i\frac{\partial \bar{f}}{\partial y}\right) = \frac{1}{2}(u_x - iv_x - i(u_y - iv_y)).\end{aligned} \tag{3}$$

We substitute equations (1) to (2), multiply the second equation in (2) by i, subtract them to get

$$\begin{aligned}\frac{\partial h}{\partial z} &= \frac{\partial g}{\partial u}\left(\frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2}i\frac{\partial u}{\partial y}\right) + \frac{\partial g}{\partial v}\left(\frac{1}{2}\frac{\partial v}{\partial x} - \frac{1}{2}i\frac{\partial v}{\partial y}\right) \\ &= (g_w + g_{\bar{w}})\frac{1}{2}(u_x - iu_y) + \frac{1}{i}(g_{\bar{w}} - g_w)\frac{1}{2}(v_x - iv_y) \\ &= g_w\frac{1}{2}(u_x - iu_y + iv_x + v_y) + g_{\bar{w}}\frac{1}{2}(u_x - iu_y - iv_x - v_y) \\ &= g_w f_z + g_{\bar{w}}\bar{f}_z\end{aligned} \tag{4}$$

using equations (3). Similarly we get

$$\begin{aligned}\frac{\partial h}{\partial \bar{z}} &= \frac{\partial g}{\partial u}\left(\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{2}i\frac{\partial u}{\partial y}\right) + \frac{\partial g}{\partial v}\left(\frac{1}{2}\frac{\partial v}{\partial x} + \frac{1}{2}i\frac{\partial v}{\partial y}\right) \\ &= (g_w + g_{\bar{w}})\frac{1}{2}(u_x + iu_y) + \frac{1}{i}(g_{\bar{w}} - g_w)\frac{1}{2}(v_x + iv_y) \\ &= g_w\frac{1}{2}(u_x + iu_y + iv_x - v_y) + g_{\bar{w}}\frac{1}{2}(u_x + iu_y - iv_x + v_y) \\ &= g_w f_{\bar{z}} + g_{\bar{w}}\bar{f}_{\bar{z}},\end{aligned} \tag{5}$$

since

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}(u_x + iv_x + i(u_y + iv_y)) \\ \frac{\partial \bar{f}}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial \bar{f}}{\partial x} + i\frac{\partial \bar{f}}{\partial y}\right) = \frac{1}{2}(u_x - iv_x + i(u_y - iv_y)).\end{aligned}$$

7. $*$ Recall the polar coordinated in the plane:

$$r = |z| = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x),$$

which can be solved for x, y to give

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Show that in polar coordinates the Cauchy–Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the function $f(z) = 1/z$ is holomorphic when $z \neq 0$ and find its derivative.

In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. By the chain rule for functions in two variables we have

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}(-r)\sin\theta + \frac{\partial u}{\partial y}r\cos\theta \\
\frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta \\
\frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x}(-r)\sin\theta + \frac{\partial v}{\partial y}r\cos\theta.
\end{aligned} \tag{6}$$

Assume the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{7}$$

Then we get

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta = \frac{1}{r}\frac{\partial v}{\partial \theta}, \\
\frac{\partial u}{\partial \theta} &= \frac{\partial v}{\partial y}(-r\sin\theta) - \frac{\partial v}{\partial x}r\cos\theta = -r\frac{\partial v}{\partial r}.
\end{aligned} \tag{8}$$

Conversely assume the Cauchy-Riemann equations in polar form (8). By (6) we get

$$\begin{aligned}
\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta &= -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta \\
\frac{\partial u}{\partial x}(-r)\sin\theta + \frac{\partial u}{\partial y}r\cos\theta &= -\frac{\partial v}{\partial x}r\cos\theta + \frac{\partial v}{\partial y}(-r)\sin\theta
\end{aligned}$$

This is a system of linear equations in the unknown functions $\partial u/\partial x, \partial u/\partial y$. Cramer's rule for solving the system gives

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -v_x\sin\theta + v_y\cos\theta & \sin\theta \\ -rv_x\cos\theta - rv_y\sin\theta & r\cos\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}} = \frac{rv_y}{r} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\begin{vmatrix} \cos\theta & -v_x \sin\theta + v_y \cos\theta \\ -r\sin\theta & -rv_x\cos\theta - rv_y\sin\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}} = \frac{-rv_x}{r} = \frac{\partial v}{\partial x}.$$

So we get back the Cauchy–Riemann equations.

For the function $f(z) = 1/z = \frac{1}{r}(\cos\theta - i\sin\theta)$, where $z = r(\cos\theta + i\sin\theta)$ is the polar form of z, we have

$$\frac{\partial u}{\partial r} = \frac{\partial(r^{-1}\cos\theta)}{\partial r} = -r^{-2}\cos\theta, \quad \frac{\partial u}{\partial\theta} = -r^{-1}\sin\theta,$$
$$\frac{\partial v}{\partial r} = \frac{\partial(-r^{-1}\sin\theta)}{\partial r} = r^{-2}\sin\theta, \quad \frac{\partial v}{\partial\theta} = -r^{-1}\cos\theta.$$

From this it is obvious that the Cauchy–Riemann equations in polar form are satisfied. To compute the derivative of f, we recall that

$$f'(z) = u_x + iv_x = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x} + i\left(\frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial\theta}\frac{\partial\theta}{\partial x}\right)$$
$$= -r^{-2}\cos\theta\cos\theta - r^{-1}\sin\theta\frac{-\sin\theta}{r} + i\left(r^{-2}\sin\theta\cos\theta - r^{-1}\cos\theta\frac{-\sin\theta}{r}\right)$$
$$= -r^{-2}(\cos^2\theta - \sin^2\theta) + ir^{-2}2\sin\theta\cos\theta = -r^{-2}\cos(2\theta) + ir^{-2}\sin(2\theta)$$
$$= -r^{-2}(\cos(-2\theta) + i\sin(-2\theta)) = -z^{-2},$$

by De Moivre's formula. We have used the fact that

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\theta, \quad \frac{\partial\theta}{\partial x} = \frac{\partial\arctan(y/x)}{\partial x} = \frac{-y}{x^2+y^2} = \frac{-\sin\theta}{r}.$$

8. $*$ Let $f(z) = u(x,y) + iv(x,y)$ be holomorphic on the domain Ω. Suppose that

$$u(x,y)^2 - u(x,y)\cdot v(x,y) + v(x,y)^2$$

is constant for all $z \in \Omega$. Show that f is a constant function.

Hint: Imitate the proof for $|f|$ constant.

Here $u = \Re(f)$ and $v = \Im(f)$, so that $u(x,y), v(x,y) \in \mathbb{R}$. Assume $u^2+uv+v^2 = k$. If $k = 0$, then we solve the quadratic equation $u^2-uv+v^2 = 0$ for u. As $D = v^2-4v^2 = -3v^2 \le 0$, and u is real, we see that there is no solution, unless $v = 0 \implies u = 0$. This gives $f(z) = 0$.

The general case is $k \neq 0$. We differentiate $u^2 - uv + v^2 = k$ in x and y to get

$$\begin{aligned} 2uu_x - (u_xv + uv_x) + 2vv_x &= 0 \\ 2uu_y - (u_yv + uv_y) + 2vv_y &= 0. \end{aligned}$$

Using the Cauchy–Riemann equations in the second equation above, we get the system

$$\left.\begin{array}{ll} 2uu_x - (u_x v + uv_x) + 2vv_x & = 0 \\ -2uv_x - (-v_x v + uu_x) + 2vu_x & = 0 \end{array}\right\} \Leftrightarrow \left\{ \begin{array}{ll} (2u - v)u_x + (-u + 2v)v_x & = 0 \\ (2v - u)u_x - (2u - v)v_x & = 0 \end{array}\right. .$$

The system has a unique solution $u_x = v_x = 0$, if the determinant of the coefficients is non-zero:

$$\left| \begin{array}{cc} 2u - v & -u + 2v \\ 2v - u & -(2u - v) \end{array} \right| = -(2u - v)^2 - (-u + 2v)^2 < 0,$$

unless $2u - v = -u + 2v = 0$ and this system gives

$$u = 2v = 2(2u) = 4u \Longrightarrow u = v = 0$$

as the unique solution. But this is impossible, as $u^2 - uv + v^2 = k \neq 0$. Therefore, we conclude that $f'(z) = u_x + iv_x = 0$ and f is constant.

Math 2101

Homework 3

Due: October 21, 2013

The first four exercises of the homework deal with questions relating to lim sup and lim inf and can be solved without complex analysis.

1. $*$ Show that if $\langle a_n \rangle$ is a convergent sequence in $\mathbb{R}$ and $\lim a_n = l$, then $\limsup a_n = l$.

   Given $\epsilon > 0$ there exists a natural number N such that

   $$n > N \Longrightarrow |a_n - l| < \epsilon \Leftrightarrow l - \epsilon < a_n < l + \epsilon.$$

   Take $M > N$. This gives

   $$\forall n > M, a_n < l + \epsilon \Longrightarrow \sup_{n \geq M} a_n \leq l + \epsilon,$$

   as $l + \epsilon$ is an upper bound for the terms involved. The limit of the sequence $b_m = \sup_{n \geq m} a_n$ is not affected by excluding the first N terms. Therefore, by taking the limit as $M \to \infty$, we deduce:

   $$\limsup a_n = \lim_{M \to \infty} \sup_{n \geq M} a_n \leq l + \epsilon.$$

   As this is true for all $\epsilon > 0$, we deduce that $\limsup a_n \leq l$. Similarly we get that

   $$l - \epsilon < a_n \leq \sup_{n \geq M} a_n, \quad \forall M > N.$$

   This implies

   $$l - \epsilon \leq \lim_{M \to \infty} \sup_{n \geq M} a_n = \limsup a_n.$$

   As ϵ is arbitrary, this implies $l \leq \limsup a_n$.

2. Let a_n, b_n be real and positive and assume that the sequence $\langle b_n \rangle$ is convergent with $\lim b_n = b > 0$. Assume also $\limsup a_n = a$. Show

   $$\limsup(a_n b_n) = ab.$$

   Deduce that $\limsup \sqrt[n]{n|a_n|} = \limsup \sqrt[n]{|a_n|}$.

   Given $\epsilon > 0$ we can find a natural number N such that

   $$n > N \Longrightarrow b - \epsilon < b_n < b + \epsilon.$$

We work only with ϵ small enough that $b-\epsilon>0$, i.e. $\epsilon<b$. This is possible, as $b>0$ is given. We multiply with $a_n>0$ to get

$$(b-\epsilon)a_n < a_n b_n < a_n(b+\epsilon).$$

Now for $N_1>N$ we take supremum on all term over $n\geq N_1$. Multiplication by the positive constants $b-\epsilon$ and $b+\epsilon$ is not a problem. We deduce that

$$(b-\epsilon)\sup_{n\geq N_1} a_n \leq \sup_{n\geq N_1}(a_n b_n) \leq (b+\epsilon)\sup_{n\geq N_1} a_n.$$

Finally we take limit as $N_1\to\infty$ to deduce that

$$(b-\epsilon)\limsup a_n = (b-\epsilon)\lim_{N_1\to\infty}\sup_{n\geq N_1} a_n \leq \limsup(a_n b_n) = \lim_{N_1\to\infty}\sup_{n\geq N_1}(a_n b_n)$$

$$\leq (b+\epsilon)\lim_{N_1\to\infty}\sup_{n\geq N_1} a_n = (b+\epsilon)\limsup a_n.$$

Since ϵ was arbitrary (>0) we deduce that

$$b\limsup a_n \leq \limsup(a_n b_n) \leq b\limsup a_n,$$

i.e. $ba=\limsup(a_n b_n)$.

Since $b_n=\sqrt[n]{n}\to 1>0$, we directly see that $\limsup\sqrt[n]{n|a_n|}=\limsup\sqrt[n]{|a_n|}$.

3. Show that, $\langle a_n\rangle_{n=0}^{\infty}$ is a sequence of non-zero complex numbers, then

$$\liminf\frac{|a_{n+1}|}{|a_n|} \leq \liminf|a_n|^{1/n} \leq \limsup|a_n|^{1/n} \leq \limsup\frac{|a_{n+1}|}{|a_n|}.$$

Since we work with absolute values, we might as well prove the result for the sequence $b_n=|a_n|$. Set

$$l=\liminf\frac{b_{n+1}}{b_n},\quad L=\liminf\sqrt[n]{b_n},\quad M=\limsup\sqrt[n]{b_n},\quad m=\limsup\frac{b_{n+1}}{b_n}.$$

Fix $\epsilon>0$. Since $m=\limsup b_{n+1}/b_n=\inf_N\sup_{n\geq N} b_{n+1}/b_n$, the number $m+\epsilon$ is not a lower bound, i.e., we can find a N such that $\sup_{n\geq N} b_{n+1}/b_n<m+\epsilon$, i.e. for all $n\geq N$ we have

$$\frac{b_{n+1}}{b_n}<m+\epsilon.$$

We apply this successively for $n=N,N+1,\ldots,n-1$ to get

$$\begin{array}{rcl} b_{N+1} & < & (m+\epsilon)b_N \\ b_{N+2} & < & (m+\epsilon)b_{N+1} \\ \vdots & & \vdots \\ b_n & < & (m+\epsilon)b_{n-1}. \end{array}$$

We multiply together to get, after cancellations,

$$b_n < (m+\epsilon)^{n-N} b_N.$$

We take n-th roots to get

$$\sqrt[n]{b_n} < (m+\epsilon)\left(\frac{b_N}{(m+\epsilon)^N}\right)^{1/n}, \quad n \geq N.$$

We take lim sup of both sides, taking into account that this does not depend on a finite number of initial terms and that $c^{1/n} \to 1$ for $c > 0$. This gives:

$$\limsup \sqrt[n]{b_n} \leq (m+\epsilon).$$

Since this is true for all $\epsilon > 0$, we get $\limsup \sqrt[n]{b_n} = M \leq m$.

For the lim inf we work analogously: Fix $\epsilon > 0$. Find N such that $\inf_{n \geq N} b_{n+1}/b_n > l - \epsilon$, which implies, for $n \geq N$, $b_{n+1}/b_n > l - \epsilon$. We apply this successively for $n = N, N+1, \ldots, n-1$ to get

$$\begin{array}{rcl} b_{N+1} & > & (l-\epsilon) b_N \\ b_{N+2} & > & (l-\epsilon) b_{N+1} \\ \vdots & & \vdots \\ b_n & > & (l-\epsilon) b_{n-1}. \end{array}$$

We multiply together to get, after cancellations,

$$b_n > (l-\epsilon)^{n-N} b_N.$$

We take n-th roots to get

$$\sqrt[n]{b_n} > (l-\epsilon)\left(\frac{b_N}{(l-\epsilon)^N}\right)^{1/n}, \quad n \geq N.$$

We take lim inf of both sides, taking into account that this does not depend on a finite number of initial terms and that $c^{1/n} \to 1$ for $c > 0$. This gives:

$$\liminf \sqrt[n]{b_n} \geq (l-\epsilon).$$

Since this is true for all $\epsilon > 0$, we get $\liminf \sqrt[n]{b_n} = L \geq l$.

The middle inequality follows from $\inf_{n \geq N} \sqrt[n]{b_n} \leq \sup_{n \geq N} \sqrt[n]{b_n}$ which gives by taking the limit as $N \to \infty$

$$\liminf \sqrt[n]{b_n} \leq \limsup \sqrt[n]{b_n}.$$

Exercise 1 and the similar result for lim inf imply that, if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$,, then

$$\liminf \frac{|a_{n+1}|}{|a_n|} = L = \limsup \frac{|a_{n+1}|}{|a_n|}.$$

Using the inequalities in Exercise 3, we deduce that

$$\liminf \sqrt[n]{|a_n|} = L = \limsup \sqrt[n]{|a_n|}.$$

Now

$$\inf_{n\geq N} \sqrt[n]{|a_n|} \leq \sqrt[N]{|a_N|} \leq \sup_{n\geq N} \sqrt[n]{|a_n|}.$$

We take the limit as $N \to \infty$ and use the sandwich theorem to deduce that

$$\liminf \sqrt[n]{|a_n|} = \lim \sqrt[n]{|a_n|} = \limsup \sqrt[n]{|a_n|} = L.$$

4. $*$ Show that if $\langle a_n \rangle_{n=0}^{\infty}$ is a sequence of non-zero complex numbers such that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L,$$

then

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

This is the ratio test and it can be used for the calculation of the radius of convergence of a power series.

Exercise 1 and the similar result for lim inf imply that, if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L,$, then

$$\liminf \frac{|a_{n+1}|}{|a_n|} = L = \limsup \frac{|a_{n+1}|}{|a_n|}.$$

Using the inequalities in Exercise 3, we deduce that

$$\liminf \sqrt[n]{|a_n|} = L = \limsup \sqrt[n]{|a_n|}.$$

Now

$$\inf_{n\geq N} \sqrt[n]{|a_n|} \leq \sqrt[N]{|a_N|} \leq \sup_{n\geq N} \sqrt[n]{|a_n|}.$$

We take the limit as $N \to \infty$ and use the sandwich theorem to deduce that

$$\liminf \sqrt[n]{|a_n|} = \lim \sqrt[n]{|a_n|} = \limsup \sqrt[n]{|a_n|} = L.$$

5. Find the radius of convergence of the hypergeometric series

$$F(\alpha, \beta, \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n.$$

Here $\alpha, \beta \in \mathbb{C}$ and $\gamma \neq 0, -1, -2, \ldots$.

We use the ratio test. We must assume that the terms are nonzero. This happens as long as α and β are not negative integers (or zero) . If either is a negative integer (or zero), the power series terminates and we get a polynomial with infinite radius of convergence. We get in the general case

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|} &= \frac{|\alpha(\alpha+1)\cdots(\alpha+n-1)(\alpha+n)\beta(\beta+1)\cdots(\beta+n-1)(\beta+n)|}{|\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)|}\\ &\cdot\frac{n!|\gamma(\gamma+1)\cdots(\gamma+n-1)(\gamma+n)|}{(n+1)!|\gamma(\gamma+1)\cdots(\gamma+n-1)|} = \frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)} \to 1, n\to\infty.\end{aligned}$$

This implies that $R = 1$.

6. Show that

$$\frac{1}{z^2} = 1 + \sum_{n=1}^{\infty}(n+1)(z+1)^n$$

for $|z+1| < 1$.

We see a power series centered at -1 on the right-hand side. So we write

$$\frac{1}{z^2} = \frac{1}{(1-(z+1))^2}.$$

The right-hand side is not a geometric series but it reminds us of it. Let

$$f(w) = \frac{1}{1-w} = \sum_{n=0}^{\infty} w^n, \quad |w| < 1.$$

Then

$$f'(w) = \frac{1}{(1-w)^2} = \sum_{n=1}^{\infty} nw^{n-1}$$

by differentiating the power series termwise. We plug now $w = (1+z)$ to get

$$\frac{1}{z^2} = \sum_{n=1}^{\infty} n(z+1)^{n-1} = 1 + \sum_{n=2}^{\infty} n(z+1)^{n-1} = 1 + \sum_{m=1}^{\infty}(m+1)(z+1)^m,$$

by shifting the index of summation ($m = n-1$).

7. $*$ Show that the radius of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n} z^{n(n+1)}$$

is 1, and discuss the convergence for $z = 1, -1, i$.

Hint: The n-th coefficient of this series is not $(-1)^n/n$.

The exponents of z are the integers $2, 6, 12, 20, \ldots$. So many natural numbers are missing, i.e. $a_m = 0$ for $m \notin \{n(n+1); n \in \mathbb{N}\}$. We cannot use the ratio test, which requires to consider the quotient of successive terms a_{n+1}/a_n (we cannot divide by 0). We use instead the definition of the radius of convergence as

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}}.$$

We have

$$\lim \sqrt[n(n+1)]{|a_{n(n+1)}|} = \lim \left(\frac{1}{n}\right)^{1/(n(n+1))} = \lim e^{-\ln n/(n(n+1))} = 1$$

as the exponent has limit 0. If you are not sure why, use L'Hôpital's rule:

$$\lim \frac{\ln n}{n(n+1)} = \lim \frac{1/n}{2n+1} = 0.$$

Therefore, $\limsup \sqrt[n(n+1)]{|a_{n(n+1)}|} = 1$, by exercise 1. Now since the rest of the $a_m = 0$, they do not influence the $\sup_{n>m} \sqrt[n]{|a_n|}$ or its limit as $m \to \infty$. Therefore,

$$\limsup \sqrt[n]{|a_n|} = 1.$$

Recall the alternating series test: If $a_n \geq 0$ and $\langle a_n \rangle$ is a nonincreasing (decreasing) sequence with $\lim a_n = 0$, then the series

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges.

For $z = 1$ we get the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which by the alternating series test converges.

For $z = -1$ we notice that $n(n+1)$ is always an even number, as either n or $n+1$ is even. Therefore $(-1)^{n(n+1)} = 1$. This gives the same series as for $z = 1$ and it converges by the alternating series test.

For $z = i$ we recall that

$$i^n = \begin{cases} 1, & n \equiv 0 \pmod 4, \\ i, & n \equiv 1 \pmod 4, \\ -1, & n \equiv 2 \pmod 4, \\ -i, & n \equiv 3 \pmod 4. \end{cases}$$

Therefore,

$$i^{n(n+1)} = \begin{cases} 1, & n \equiv 0 \pmod 4, \\ -1, & n \equiv 1 \pmod 4, \\ -1, & n \equiv 2 \pmod 4, \\ 1, & n \equiv 3 \pmod 4. \end{cases}$$

since $n(n+1)$ is divisible by 4, when $n \equiv 0$ or $3 \pmod 4$ and in the other two cases $n(n+1) \equiv 2 \pmod 4$. So the series becomes

$$T = \frac{1}{1} - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \cdots$$

The pattern is that after the first term, we have two $-$ followed by two $+$, followed by two $-$ and so on. We claim that this series converges. Look at essentially the same series written as

$$S = \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{2n} + \frac{1}{2n+1} \right) = -\left(\frac{1}{2} + \frac{1}{3} \right) + \left(\frac{1}{4} + \frac{1}{5} \right) - \cdots$$

Since

$$\frac{1}{2n} + \frac{1}{2n+1} = \frac{4n+1}{2n(2n+1)} \to 0$$

and is decreasing, the alternating series test shows that S converges. The partial sums T_n of T differ from the partial sums S_n of S as follows:

$$T_{2n} = \frac{1}{1} - \frac{1}{2} - \frac{1}{3} + \cdots \pm \frac{1}{2n} = 1 + S_n \mp \frac{1}{2n+1},$$

$$T_{2n+1} = \frac{1}{1} - \frac{1}{2} - \frac{1}{3} + \cdots \mp \frac{1}{2n+1} = 1 + S_n.$$

Therefore, the partial sums T_n converge to the sum $S + 1$ and the series T converges.

8. Prove that, although all the following power series have radius of convergence $R = 1$,

   (a) $\sum nz^n$ does not converge on any point of the unit circle $\{z, |z| = 1\}$,

   (b) $\sum z^n/n^2$ converges at every point of the unit circle.

   Since $\lim \sqrt[n]{n} = 1$ it is easy to see that $R = 1$. For (a) we notice that on $|z| = 1$, $|nz^n| = n \to \infty$, so the series cannot converge, as the general term does not go to 0. For (b) we notice that we can apply the comparison test with $b_n = n^{-2}$, where $\sum b_n < \infty$. We have on $|z| = 1$ the bound $|z^n/n^2| = b_n$. The comparison test implies that the series converges.

9. $*$ The Fibonacci numbers are defined by $f_0 = 1$, $f_1 = 1$,

$$f_n = f_{n-1} + f_{n-2}, \quad n = 2, 3, \ldots.$$

Define their generating function as

$$F(z) = \sum_{n=0}^{\infty} f_n z^n.$$

(a) Find a quadratic polynomial $Az^2 + Bz + C$ such that

$$(Az^2 + Bz + C)F(z) = 1.$$

We get

$$\sum_{n=0}^{\infty} Af_n z^{n+2} + \sum_{n=0}^{\infty} Bf_n z^{n+1} + \sum_{n=0}^{\infty} Cf_n z^n = 1,$$

$$\sum_{n=2}^{\infty} Af_{n-2} z^n + \sum_{n=1}^{\infty} Bf_{n-1} z^n + \sum_{n=0}^{\infty} Cf_n z^n = 1,$$

$$\sum_{n=2}^{\infty}(Af_{n-2} + Bf_{n-1} + Cf_n)z^n + (Bf_0 + Cf_1)z + Cf_0 = 1.$$

By the uniqueness of the coefficients of the power series, we get

$$Af_{n-2} + Bf_{n-1} + Cf_n = 0, \quad Bf_0 + Cf_1 = 0, \quad Cf_0 = 1.$$

Using $f_0 = f_1 = 1$ we get $C = 1$, $B = -1$, while the first equation becomes $Af_{n-2} - f_{n-1} + f_n = 0$. The recurrence formula $f_n = f_{n-1} + f_{n-2}$ implies $A = -1$. So the quadratic polynomial is $-z^2 - z + 1$.

(b) Use partial fractions to determine the following closed expression for f_n.

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}.$$

The roots of the quadratic polynomial are

$$\rho_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$$

with $\rho_1\rho_2 = -1$. We set

$$\frac{1}{-z^2 - z + 1} = \frac{A}{z - \rho_1} + \frac{B}{z - \rho_2}$$

to get

$$-1 = A(z - \rho_2) + B(z - \rho_1) \Longrightarrow A = \frac{1}{\rho_2 - \rho_1}, \quad B = \frac{1}{\rho_1 - \rho_2} = -A.$$

This gives $A = -1/\sqrt{5}$, $B = 1/\sqrt{5}$. We expand the fractions into geometric series valid for $|z/\rho_1| < 1$ and $|z/\rho_2| < 1$ respectively to get

$$\frac{A}{z-\rho_1} = \frac{1}{\sqrt{5}} \frac{1}{\rho_1(1-z/\rho_1)} = \frac{1}{\sqrt{5}} \sum_0^\infty \frac{z^n}{\rho_1^{n+1}},$$

and

$$\frac{B}{z-\rho_1} = \frac{-1}{\sqrt{5}} \frac{1}{\rho_2(1-z/\rho_2)} = \frac{-1}{\sqrt{5}} \sum_0^\infty \frac{z^n}{\rho_2^{n+1}}.$$

This gives, by the uniqueness of the power series

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1}{\rho_1^{n+1}} - \frac{1}{\rho_2^{n+1}} \right)$$

$$f_n = \frac{1}{\sqrt{5}} \left((-\rho_2)^{n+1} - (-\rho_1)^{n+1} \right)$$

using $\rho_1 \rho_2 = -1$.

10. (i) Show that the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

can be calculated as

$$\Delta = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z}.$$

(ii) Show that for any analytic function $f(z)$ we have

$$\Delta |f(z)|^2 = 4|f'(z)|^2.$$

(i) We have for a function f with continuous second partial derivatives

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$$

which gives

$$4 \frac{\partial^2 f}{\partial z \partial \bar{z}} = \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (f_x + i f_y) = \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f_x + i \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f_y$$
$$= f_{xx} - i f_{yx} + i(f_{xy} - i f_{yy}) = f_{xx} - i f_{yx} + i f_{xy} + f_{yy} = f_{xx} + f_{yy} = \Delta f,$$

since the mixed partial f_{xy}, f_{yx} are equal for functions with continuous second partial derivatives. Reversing the order of the calculation gives $4\partial_{\bar{z}} \partial_z f = \Delta f$.

(ii) For an analytic function $\partial_z f(z) = f'(z)$, while $\partial_{\bar{z}} \overline{f(z)} = \overline{f'(z)}$, since f does not depend on $\bar{z}$, so the result from applying $\partial_{\bar{z}}$ on its conjugate can be calculated from $f'(z)$: e.g. $f(z) = z^2$, $\overline{f(z)} = \bar{z}^2$,$\partial_{\bar{z}} \overline{f(z)} = 2\bar{z} = \overline{2z} = \overline{\partial_z f(z)}$. This gives:

$$\Delta |f(z)|^2 = 4\partial_z \partial_{\bar{z}} (f(z)\overline{f(z)}) = 4\partial_z (f(z)\overline{f'(z)}) = 4\overline{f'(z)} \partial_z f(z) = 4\overline{f'(z)} f'(z) = 4|f'(z)|^2,$$

since $f(z)$ is a constant when we differentiate in $\bar{z}$, and $\overline{f'(z)}$ is a constant when we differentiate in z, as it does not depend on z.

Math 2101

Homework 4

Due: October 28, 2013

1. (a) Show that $e^{\bar{z}} = \overline{e^z}$.

   Let $z = x + iy$. Since $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$, we get

$$\overline{e^z} = e^x(\cos y - i\sin y) = e^x(\cos(-y) + i\sin(-y)) = e^x e^{-iy} = e^{x-iy} = e^{\bar{z}}.$$

   (b) Show that $e^{\bar{z}}$ is not holomorphic at any point in $\mathbb{C}$.

   We have that $u = \Re(e^{\bar{z}}) = e^x \cos y$ and $v = \Im(e^{\bar{z}}) = -e^x \sin y$. The first Cauchy–Riemann equation means

$$e^x \cos y = u_x = v_y = -e^x \cos y \Longrightarrow \cos y = 0.$$

   The second Cauchy–Riemann equation means

$$-e^x \sin y = u_y = -v_x = e^x \sin y \Longrightarrow \sin y = 0.$$

   There is no $y \in \mathbb{R}$ with $\sin y = \cos y = 0$, by the basic trigonometric identity $\cos^2 t + \sin^2 t = 1$.

   (c) Find the image of the semi-infinite strip $x \geq 0$ and $0 \leq y \leq \pi$ under the transformation $w = e^z$. Exhibit corresponding portions of the boundaries.

   As usual we set $w = u + iv$. Since $e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$ has argument y and modulus e^x, we see that the horizontal rays $y = c$, $x > 0$ are mapped to the rays given in polar coordinates $r = e^x$, $x > 0$, $\theta = c$. Since $e^x > 1$ we get rays emanating at the circle of radius 1. As we vary y between 0 and π we get all rays with angle from the positive real axis between 0 and π. These rays cover the upper half-plane with the half-disk $|w| < 1$ removed. For the positive real axis given by $x > 0$ and $y = 0$ we get as image the ray $u > 1$ and $v = 0$. For the segment on the imaginary axis given by $x = 0$ and $0 \leq y \leq \pi$ we get the semicircle $|w| = 1$ and $v \geq 0$. For the ray given by $y = \pi$ and $x > 0$ we get the ray $u = -e^x$ i.e. $u < -1$ and $v = 0$.

2. $*$ For $z = x + iy$ show that

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y).$$

   Deduce that

$$|\cos(z)|^2 = \cos^2 x + \sinh^2(y),$$

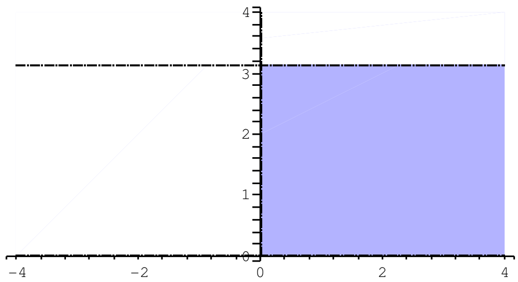


Figure 1: The region for 1(c) in the z plane in blue

and

$$|\sinh(y)| \leq |\cos(z)| \leq \cosh(y).$$

Recall that

$$\cosh y = \frac{e^y + e^{-y}}{2}, \quad \sinh y = \frac{e^y - e^{-y}}{2},$$

so that

$$e^y = \cosh y + \sinh y, \quad e^{-y} = \cosh y - \sinh y.$$

We have

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{2}$$

$$= \frac{e^{ix}\cosh y - e^{ix}\sinh y + e^{-ix}\cosh y + e^{-ix}\sinh y}{2} = \frac{e^{ix} + e^{-ix}}{2}\cosh y - i\frac{e^{ix} - e^{-ix}}{2i}\sinh y$$

$$= \cos(x)\cosh y - i\sin(x)\sinh y.$$

For the modulus we use the identity $\cosh^2(y) - \sinh^2(y) = 1$ to get

$$|\cos(z)|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = \cos^2 x(1 + \sinh^2 y) + \sin^2 x \sinh^2 y$$

$$= \cos^2 x + \cos^2 x \sinh^2 y + \sin^2 x \sinh^2 y = \cos^2 x + (\sin^2 x + \cos^2 x)\sinh^2 y = \cos^2 x + \sinh^2 y.$$

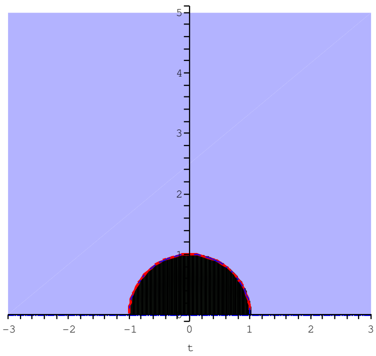


Figure 2: The region for 1(c) in the w plane in blue. The black half-disc corresponds to the left semi-infinite horizontal strip.

By bounding from above the $\cos^2 x$ by 1 we see that

$$|\cos(z)|^2 \leq 1 + \sinh^2 y = \cosh^2 y \Longrightarrow |\cos(z)| \leq \cosh(y).$$

The left inequality is even easier since $\cos^2(x) \geq 0$. We get

$$|\cos(z)|^2 = \cos^2 x + \sinh^2 y \geq \sinh^2 y \Longrightarrow |\cos(z)| \geq |\sinh y|.$$

3. $*$ Solve the equation $\cosh(z) = 1/2$.

   We have $\cosh z = (e^z + e^{-z})/2$ so that

$$\cosh(z) = \frac{1}{2} \Leftrightarrow \frac{e^z + e^{-z}}{2} = \frac{1}{2} \Leftrightarrow e^z + e^{-z} = 1 \Leftrightarrow e^{2z} + 1 = e^z,$$

   as $e^z \neq 0$. We rewrite the equation to see that it is a quadratic equation with unknown e^z, so that we can set $w = e^z$:

$$(e^z)^2 - e^z + 1 = 0 \Leftrightarrow w^2 - w + 1 = 0 \Leftrightarrow w = \frac{1 \pm i\sqrt{3}}{2} = \cos(\pm\pi/3) + i\sin(\pm\pi/3) = e^{\pm\pi i/3}.$$

   This gives

$$e^z = e^{\pm\pi i/3} \Leftrightarrow z = \pm\pi/3 + 2n\pi i, \quad n \in \mathbb{Z}.$$

4. Denote by Log (z) the principal logarithm. Show that if $\Re(z_1) > 0$ and $\Re(z_2) > 0$, then

$$\text{Log }(z_1 z_2) = \text{Log }(z_1) + \text{Log }(z_2).$$

   Show that this formula is not true in general by founding a counterexample.

Recall that the principal logarithm of z is defined as

$$\text{Log } z = \log|z| + i\text{Arg }(z), \quad \text{Arg }(z) \in (-\pi, \pi].$$

If z_i satisfy $\Re(z_i) > 0$, the principal arguments Arg (z_i) are in the interval $(-\pi/2, \pi/2)$. We know that

$$\log|z_1 z_2| = \log(|z_1||z_2|) = \log|z_1| + \log|z_2|$$

by the familiar properties of the standard logarithms, as here $|z_1|$ and $|z_2|$ are real. As far as the principal argument of $z_1 z_2$, we choose the argument of $z_1 z_2$ in the range $(-\pi, \pi]$. As Arg z_1 + Arg $z_2 \in (-\pi, \pi)$ and the argument of the product can be taken to be the sum of the arguments ($+2\pi i k$, $k \in \mathbb{Z}$), we get

$$\text{Arg }(z_1 z_2) = \text{Arg } z_1 + \text{Arg }(z_2).$$

By summing the real and imaginary parts, the result is the identity:

$$\text{Log }(z_1 z_2) = \text{Log }(z_1) + \text{Log }(z_2).$$

In general the formula is not correct and it suffices to choose two complex numbers with principal arguments summing up to an angle $> \pi$. E.g. we can take $z_1 = i$, $z_2 = -1$. Then

$$\text{Log }(z_1) = \text{Log }(i) = \log|i| + i\pi/2 = \log 1 + i\pi/2 = i\pi/2.$$

$$\text{Log }(z_2) = \log|-1| + i\pi = i\pi.$$

These give

$$\text{Log }(z_1 z_2) = 3\pi i/2.$$

On the other hand $z_1 z_2 = -i$.

$$\text{Log }(-i) = \log|-i| + i(-\pi/2) = -i\pi/2.$$

5. Show that the transformation $w = z^2$ maps the lines $x = c$ for $c \neq 0$ onto the parabolas $v^2 = -4c^2(u - c^2)$ and the lines $y = d$ for $d \neq 0$ onto the parabolas $v^2 = 4d^2(u + d^2)$. Prove that at a point of intersection the parabolas meet orthogonally in two ways: (a) directly and (b) quoting a theorem.

   We have $w = u + iv = z^2 = (x^2 - y^2) + 2ixy \Longrightarrow u = x^2 - y^2, v = 2xy$. With $x = c$ we have

$$v = 2cy, \quad u = c^2 - y^2 = c^2 - (v/(2c))^2 = c^2 - \frac{v^2}{4c^2} \Longrightarrow 4c^2 u = 4c^4 - v^2 \Longrightarrow$$

$$v^2 = 4c^4 - 4c^2 u = -4c^2(u - c^2).$$

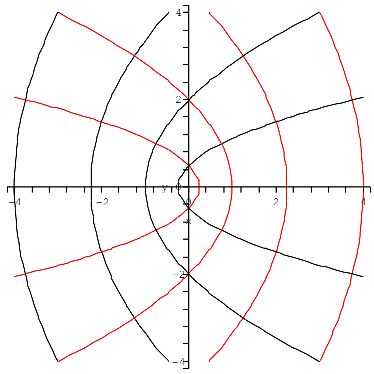


Figure 3: The parabolas in problem 6 in red for values $c = 1/2, 1, 3/2, 2$ and in black for values $d = 1/2, 1, 3/2, 2$

Similarly with $y = d$

$$v = 2xd, \quad u = x^2 - d^2 = (v/(2d))^2 - d^2 = \frac{v^2}{4d^2} - d^2 \Longrightarrow 4d^2u = v^2 - 4d^4 \Longrightarrow$$
$$v^2 = 4d^2(u + d^2).$$

(a) We use implicit differentiation in the equations of the parabolas (v is dependent variable and u is independent), and we work at a common point of the two parabolas. For the first we calculate the slope of the tangent line as follows:

$$2vv' = -4c^2 \Longrightarrow v' = -\frac{4c^2}{2v} = -\frac{2c^2}{v}.$$

On the other hand for the second parabola

$$2vv' = 4d^2 \Longrightarrow v' = \frac{4d^2}{2v} = \frac{2d^2}{v}.$$

To show that the tangent lines are perpendicular we must show that the product of the slopes is -1, i.e.

$$-\frac{2c^2}{v} \cdot \frac{2d^2}{v} = -1 \Leftrightarrow v^2 = 4c^2d^2.$$

At the common point of the parabolas we have

$$v = 4d^2u + 4d^4 = -4c^2u + 4c^4 \Longrightarrow 4(c^2 + d^2)u = 4(c^4 - d^4) \Longrightarrow u = c^2 - d^2.$$

This gives

$$v^2 = 4d^2(c^2 - d^2) + 4d^4 = 4d^2c^2.$$

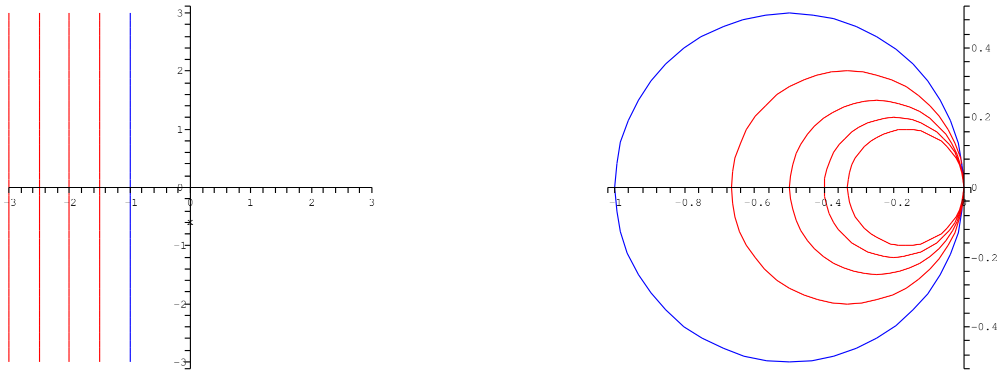


Figure 4: The region $\Re(z) < -1$ in problem 7 and its image by $w = 1/z$

(b) Since the lines $x = c$ and $y = d$ are orthogonal (perpendicular to each other), the fact that $w = z^2$ is conformal for $z \neq 0$, which follows from the fact that $dw/dz = 2z \neq 0$ and w is holomorphic, implies that the images of these curves will also meet orthogonally.

6. $*$ Show that if $c_1 < 0$, the image of the half plane with equation $x < c_1$ under the transformation $w = 1/z$ is an open disc. Write the equation of the circle bounding this disc.

The half-plane can be swept out by the vertical lines with equations $\Re(z) = k$, where $k < c_1$. We find the image of this line under $w = 1/z$. We know that it has to be a line or a circle. Since $\Re(z) = k$ does not pass through the origin, it has to be a circle $(1/0 = \infty)$. To determine the circle we see that $z = k \implies 1/z = 1/k$, while the point at infinity of the line should be mapped to 0 since $1/\infty = 0$. So the circle contains 0 and $1/k$. Moreover, the original line is perpendicular to the real axis, which is mapped to itself. The image i.e. the circle should be perpendicular to the real axis at the points of intersection. This leads to a circle with diameter on the real axis. The centre is at the midpoint from 0 to $1/k$, i.e. at $1/(2k)$. The equation is

$$(u - \frac{1}{2k})^2 + v^2 = \frac{1}{4k^2}.$$

In particular when $k = c_1$ we get the circle

$$(u - \frac{1}{2c_1})^2 + v^2 = \frac{1}{4c_1^2}. \tag{1}$$

We notice that as k decreases from c_1 to $-\infty$, the centers get closer to the origin and the radii tend to 0. All these circles cover the open disk bounded by (1).

There is another way of determining the above equations. With $w = 1/z$ we have

$$u + iv = \frac{\Re z}{|z|^2} - i\frac{\Im z}{|z|^2} = \frac{k}{k^2+y^2} - i\frac{y}{k^2+y^2}.$$

We eliminate y: we notice that

$$u^2 + v^2 = \frac{k^2+y^2}{(k^2+y^2)^2} = \frac{1}{k^2+y^2} = \frac{u}{k}$$

so that

$$0 = u^2 + v^2 - \frac{u}{k} = \left(u - \frac{1}{2k}\right)^2 - \frac{1}{4k^2} + v^2 \Longrightarrow \left(u - \frac{1}{2k}\right)^2 + v^2 = \frac{1}{4k^2}.$$

7. Let $w = \dfrac{z-1}{z+1}$. Show that it maps the right-half plane $\{z, \Re(z) > 0\}$ conformally onto the unit disc $\{z, |z| < 1\}$. Notice that the map must be bijective.

We prove that $|w| < 1 \Leftrightarrow \Re(z) > 0$.

$$|w| = \left|\frac{z-1}{z+1}\right| < 1 \Leftrightarrow |z-1| < |z+1| \Leftrightarrow |z-1|^2 < |z+1|^2$$

$$\Leftrightarrow (z-1)(\bar{z}-1) < (z+1)(\bar{z}+1) \Leftrightarrow |z|^2 - z - \bar{z} + 1 < |z|^2 + z + \bar{z} + 1 \Leftrightarrow -2\Re(z) < 2\Re(z) \Leftrightarrow \Re(z) > 0.$$

The map is holomorphic as -1 is not in the right-half plane and its inverse is given by

$$z = \frac{w+1}{1-w}$$

which is holomorphic when $|w| < 1$. Another way to see that w maps the right half-plane to $|w| < 1$ is as follows: $|z-1|$ is the distance of z to 1 and $|z+1|$ to -1. So $|w| < 1$ means that the distance from z to 1 is less than the distance from z to -1. Such points lie to the one side of the perpendicular bisector of the segment $[-1, 1]$ closer to the point 1. The bisector is clearly the imaginary axis, so we get the points to the right of it.

8. $*$ (a) Show that the map $w = \dfrac{z-i}{z+i}$ maps the upper half-plane $\{z; \Im(z) > 0\}$ conformally onto the unit disc $\{z; |z| < 1\}$.

We prove that

$$|w| < 1 \Leftrightarrow \Im(z) > 0. \tag{2}$$

$$|w| = \left|\frac{z-i}{z+i}\right| < 1 \Leftrightarrow |z-i| < |z+i| \Leftrightarrow |z-i|^2 < |z+i|^2$$

$$\Leftrightarrow (z-i)(\bar{z}+i) < (z+i)(\bar{z}-i) \Leftrightarrow z\bar{z} - i\bar{z} + iz + 1 < z\bar{z} + i\bar{z} - iz + 1$$

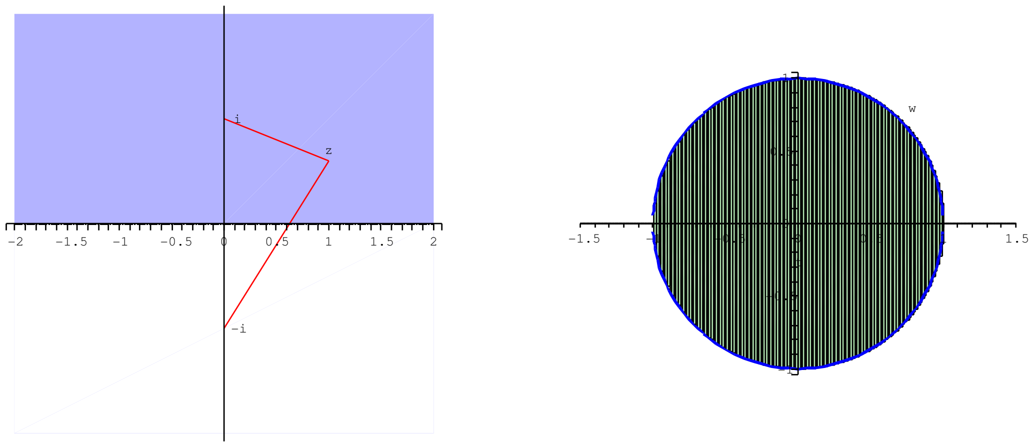


Figure 5: Regions for problem 8(a)

$$i(z-\bar{z}) < -i(z-\bar{z}) \Leftrightarrow i2i\Im(z) < -i2i\Im(z) \Leftrightarrow -2\Im(z) < 2\Im(z) \Leftrightarrow 0 < 4\Im(z) \Leftrightarrow \Im(z) > 0.$$

Geometrically the distance to i is less than the distance to $-i$ iff z is in the half-plane determined by the perpendicular bisector of the segment from i to $-i$ and containing i. The bisector is clearly the real axis.

The map is holomorphic on $\{z, \Im(z) > 0\}$ as $-i$ (root of the denominator) is not in it. It is a rational function. The inverse map is given by

$$w = \frac{z-i}{z+i} \Leftrightarrow wz + iw = z - i \Leftrightarrow z(w-1) = -i - iw = -i(w+1) \Leftrightarrow z = -i\frac{w+1}{w-1}.$$

This is also holomorphic on $D(0,1)$ as it is a rational function and $1 \notin D(0,1)$. The last calculation shows that the map is a injection. The surjectivity follows from (2).

(b) Find a conformal map from $\{z, \Re(z) > 0 \text{ and } \Im(z) > 0\}$ onto the unit disc $\{z; |z| < 1\}$. *Hint*: To start use z^2.

The map$\zeta = z^2$ maps sectors at the origin to sectors at the origin double in size: if $z = re^{i\theta}$, then

$$\zeta = r^2 e^{2i\theta}.$$

In particular we see that it maps $\{z, \Re(z) > 0 \text{ and } \Im(z) > 0\}$ to the upper-half plane $\{\zeta; \Im(\zeta) > 0\}$. Then we compose with the map from (a), which maps the upper-half plane to the unit disc. The final result is the map:

$$w = \frac{z^2 - i}{z^2 + i}.$$

9. Show that any transformation of the form

$$w = e^{i\theta}\frac{z - z_0}{1 - \bar{z}_0 \cdot z}, \quad \theta \in \mathbb{R}, \quad |z_0| < 1$$

maps the disk $|z| \leq 1$ conformally into the disc $|w| \leq 1$.

Assume that $|z| = 1$, which is the unit circle. We will prove that it is mapped to the unit circle $|w| = 1$. We remark that $|e^{i\theta}| = 1$ The denominator can be written as

$$1 - \bar{z}_0 z = z\bar{z} - \bar{z}_0 z = z(\bar{z} - \bar{z}_0) = z(\overline{z - z_0})$$

As complex conjugates have the same modulus, we have

$$|1 - \bar{z}_0 z| = |z||\overline{z - z_0}| = 1 \cdot |z - z_0| \Longrightarrow \left|\frac{z - z_0}{1 - \bar{z}_0 \cdot z}\right| = 1.$$

The argument is reversible: if $|w| = 1$ then $|z| = 1$:

$|z - z_0| = |1 - \bar{z}_0 z| \Leftrightarrow (z - z_0)(\bar{z} - \bar{z}_0) = (1 - \bar{z}_0 z)(1 - z_0\bar{z}) \Leftrightarrow |z|^2 - z_0\bar{z} - z\bar{z}_0 + |z_0|^2$

$= 1 - z_0\bar{z} - z\bar{z}_0 + |z_0|^2|z|^2 \Leftrightarrow |z|^2 + |z_0|^2 - |z_0|^2|z|^2 - 1 = 0 \Leftrightarrow (|z|^2 - 1)(|z_0|^2 - 1) = 0.$

This gives $|z| = 1$, as $|z_0| < 1$.

Since w is a linear fractional transformation, it is bijective in the (extended) complex plane $\mathbb{C} \cup \{\infty\}$. In our case we do not have to worry about ∞, as the denominator is never 0: $1 - \bar{z}_0 z = 0 \Longrightarrow z = 1/\bar{z}_0$. However, $|z_0| < 1$, while we only work with $|z| \leq 1$.

The difficulty is to show that the interior of the unit disc given by $|z| < 1$ is mapped to the interior of the disc $|w| < 1$. We know that z_0 is mapped to 0. Let z_1 have $|z_1| < 1$. We can join z_1 with z_0 by a line segment S interior to the unit disk. The image of S is either a segment on a line or an arc on a circle. If z_1 is mapped outside the unit disk, this arc or segment will meet the unit disk $|w| = 1$. The point of intersection violates the fact that the linear fractional transformation is injective. It is the image of a point on the circle $|z| = 1$ and of some point z' with $|z'| < 1$.

In the same spirit, if z_2 is outside the unit disk, then it is mapped outside i.e. to some point w_2 with $|w_2| > 1$. Otherwise $|w_2| < 1$. Join w_2 and 0 by a line segment. It must be the image of an arc or segment joining z_2 and z_0. This will intersect $|z| = 1$. The point of intersection violates the fact that the linear fractional transformation is a function. It maps to a point inside the circle $|w| = 1$ and to some point w' with $|w'| = 1$, on the circle.

Math 2101

Homework 5

Due: November 11, 2013

1. $*$ Evaluate the integrals

$$I_j = \int_{\gamma_j} (\bar{z})^2 \, dz, \quad j = 1, 2, 3,$$

where

- γ_1 is the segment from 0 to $1 + i$,
- γ_2 is the circular arc from $1 + i$ to $i\sqrt{2}$ on the circle $|z| = \sqrt{2}$, traversed anti-clockwise,
- γ_3 is the segment from $i\sqrt{2}$ to 0.

Hence determine the value of the integral

$$\int_{\gamma_1+\gamma_2+\gamma_3} (\bar{z})^2 \, dz.$$

(a) We use the parametrisation $\gamma_1(t) = t + it$, $0 \leq t \leq 1$, so that $\gamma_1'(t) = (1+i)$. Then

$$I_1 = \int_{\gamma_1} (\bar{z})^2 \, dz = \int_0^1 (\overline{t + it})^2 (1+i) \, dt = \int_0^1 (t - it)^2 (1+i) \, dt = (1+i) \int_0^1 (t^2 - 2it^2 - t^2) \, dt$$

$$= -2i(1 + i) \int_0^1 t^2 \, dt = (1 - i)\frac{2}{3}.$$

(b) We use the parametrisation $\gamma_2(t) = \sqrt{2}e^{it}$, with $\pi/4 \leq t \leq \pi/2$. Here we notice that t measures the principal argument along the circular arc and that

$$\operatorname{Arg}(1 + i) = \pi/4, \quad \operatorname{Arg}(\sqrt{2}i) = \pi/2.$$

We have $\gamma_2'(t) = \sqrt{2}ie^{it}$, while $\overline{\gamma_2(t)} = \sqrt{2}e^{-it}$, so that

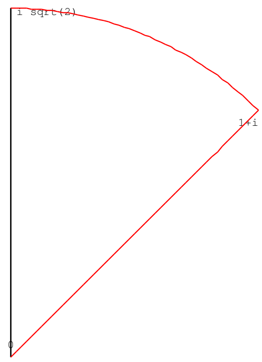
$$I_2 = \int_{\gamma_2} (\bar{z})^2 \, dz = \int_{\pi/4}^{\pi/2} \overline{\gamma_2(t)}^2 \gamma_2'(t) \, dt = \int_{\pi/4}^{\pi/2} 2e^{-2it} \sqrt{2}ie^{it} \, dt = 2\sqrt{2}i \int_{\pi/4}^{\pi/2} e^{-it} \, dt$$

$$= \frac{2\sqrt{2}i}{-i} [e^{-it}]_{\pi/4}^{\pi/2} = -2\sqrt{2}(e^{-i\pi/2} - e^{-i\pi/4}) = -2\sqrt{2}\left(-i - \frac{1 - i}{\sqrt{2}}\right) 2 + i2(\sqrt{2} - 1).$$

(c) We use the reverse curve $\tilde{\gamma}_3$ with parametrisation $\tilde{\gamma}_3(t) = it$, $0 \le t \le \sqrt{2}$. We have $\tilde{\gamma}_3'(t) = i$ so that

$$I_3 = \int_{\gamma_3} (\bar{z})^2\, dz = -\int_{-\gamma_3} (\bar{z})^2\, dz = -\int_{\tilde{\gamma}_3} (\bar{z})^2\, dz = -\int_0^{\sqrt{2}} (-it)^2 i\, dt = -\int_0^{\sqrt{2}} -t^2 i\, dt$$

$$= \frac{i}{3}[t^3]_0^{\sqrt{2}} = \frac{i2\sqrt{2}}{3}.$$



2. $*$ Compute the integral

$$\int_{|z|=r} y\, dz$$

for the circle traversed anticlockwise, in two ways: first by using a parametrisation, and second, by observing that $y = (1/(2i))(z - \bar{z}) = (1/(2i))(z - r^2/z)$ on the circle.

First Method: Parametrization of the circle as $z = re^{i\theta}$, $y = r\sin\theta$, $dz = rie^{i\theta}$. We get

$$\int_{|z|=r} y\, dz = \int_0^{2\pi} r\sin\theta r i e^{i\theta}\, d\theta = \int_0^{2\pi} r^2 i \sin\theta(\cos\theta + i\sin\theta) d\theta = \int_0^{2\pi} r^2 i(\sin\theta\cos\theta + i\sin^2\theta) d\theta$$

$$= r^2 i \int_0^{2\pi} \frac{\sin(2\theta)}{2} + \frac{i}{2}(1 - \cos(2\theta)) d\theta = r^2 i \left[-\frac{\cos 2\theta}{4} + \frac{i}{2}\theta + \frac{-i\sin 2\theta}{4}\right]_0^{2\pi} = -r^2\pi.$$

Second method: On the circle $|z| = r$, we have $z\bar{z} = r^2 \implies \bar{z} = r^2/z$, so that $y = (z - \bar{z})/(2i) = (1/(2i))(z - r^2/z)$. We plug into the integral to get

$$\int_{|z|=r} y\, dz = \int_{|z|=r} \frac{z}{2i} - \frac{r^2}{2iz}\, dz = -\frac{r^2}{2i} 2\pi i = -r^2\pi,$$

since we know that $\int_{|z|=r} z^n\, dz = 0$ for $n \neq -1$ and for $n = -1$ we get $2\pi i$.

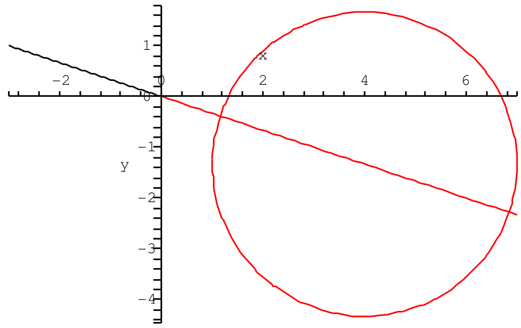


Figure 1: The black ray is outside the circle

3. In the following you are not allowed to use the residue theorem or Cauchy's integral formula, as they have not been discussed yet.

   (a) Evaluate the integrals

   $$\int_\gamma z^n \, dz$$

   for all integers n, where γ is any circle not containing the origin, tranversed in the positive sense.

   For $n \neq -1$ the function z^n has antiderivative $z^{n+1}/(n+1)$, so evaluating over the closed curve, we get that the integral is 0.

   For $n = -1$ this is not the case, as $1/z$ has no holomorphic primitive on $\mathbb{C} \setminus \{0\}$. However, every circle not containing the origin avoids at least one ray emanating through the origin: e.g. taking the line from the centre of the circle to 0. The ray in the opposite direction does not meet the circle or its interior. Let θ_0 by the angle this ray forms with the positive real axes, taken in $(-\pi, \pi]$. We can define an antiderivative of $1/z$ in the plane cut by this ray, given by the equation $\arg(z) = \theta_0$, i.e.

   $$\log(z) = \log|z| + i \arg(z), \quad \text{with } \arg(z) \in (\theta_0 - 2\pi, \theta_0)$$

   On this cut plane we easily verify that $\log(z)$ defined above is holomorphic and its derivative is $1/z$. Therefore $1/z$ has a primitive in the region containing the circle.

As the circle C is a closed curve, the antiderivative theorem gives

$$\int_C \frac{1}{z}\, dz = 0.$$

(b) Let a, b, r be numbers with $0 < a < r < b$ and let γ be the circle of radius r centered at the origin with positive orientation. Show that

$$\int_\gamma \frac{1}{(z-a)(z-b)}\, dz = \frac{2\pi i}{a-b}.$$

We write the integrand using partial fractions. We easily verify that

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left(\frac{1}{z-a} - \frac{1}{z-b}\right).$$

For

$$\int_\gamma \frac{1}{z-b} dz$$

where the contour does not contain the point b, where the denominator vanish, we can use the argument in (b) to define the primitive

$$\log(z-b) = \log|z-b| + i\arg(z-b), \quad \arg(z-b) \in (\theta_0 - 2\pi, \theta_0)$$

for appropriate choice of θ_0. Therefore,

$$\int_\gamma \frac{1}{z-b} dz = 0.$$

We need to show that

$$\int_\gamma \frac{1}{z-a} dz = 2\pi i.$$

Here the circle enclosed a but a is not the center of the circle. We argue as follows. If γ_ϵ is the curve cut on the circle γ by the horizontal lines $y = \pm\epsilon$ and containing $r > 0$, then γ_ϵ misses the negative real axis for all ϵ. Let $z_{1,\epsilon}$, $z_{2,\epsilon}$ be the starting point and endpoint of γ_ϵ. We can use the principal logarithm Log $(z-a)$ in this region, which is the primitive of $1/(z-a)$. We get

$$\int_{\gamma_\epsilon} \frac{1}{z-a} dz = \text{Log }(z_{2,\epsilon}-a) - \text{Log }(z_{1,\epsilon}-a) = \log|z_{2,\epsilon}-a| - \log|z_{1,\epsilon}-a| + i\text{Arg }(z_{2,\epsilon}-a) - i\text{Arg }(z_{1,\epsilon}-a).$$

As $\epsilon \to 0$, $z_{1,\epsilon} \to -r$ and $z_{2,\epsilon} \to -r$, so that

$$\lim_{\epsilon\to 0}(\log|z_{2,\epsilon}-a| - \log|z_{1,\epsilon}-a|) = 0.$$

However, the principal arguments tend to π and $-\pi$.

$$\lim_{\epsilon\to 0}(\text{Arg }(z_{2,\epsilon}-a)-i\text{Arg }(z_{1,\epsilon}-a))=2\pi.$$

We deduce by the fact that the integral is continuous in its limits of integration:

$$\lim_{\epsilon\to 0}\int_{\gamma_\epsilon}\frac{1}{z-a}dz=\int_{\gamma}\frac{1}{z-a}dz=2\pi i.$$

(c) Compute the integral

$$\int_{|z|=2}\frac{dz}{z^2-1}$$

for the positive sense of the circle. *Hint*: Find a primitive function of the integrand.

We use partial fractions to find that

$$\frac{1}{z^2-1}=\frac{1}{2}\left(\frac{1}{z-1}-\frac{1}{z+1}\right).$$

We would like to integrate to get $(\log(z-1)-\log(z+1))/2$. However, this cannot make sense in the region we are working: The $\log(z-1)$ requires a cut from, say $-\infty$ to 1 and $\log(z+1)$ requires a cut from $-\infty$ to -1. These cuts meet our domain, since the circle $|z|=2$ contains both points -1 and 1. Make sense can be thought of being (single-valued) holomorphic functions that give us the primitive of the terms we consider. However, if we write the expected answer as

$$F(z)=\frac{1}{2}\log\left(\frac{z-1}{z+1}\right)$$

things are much better: First of all, $(z-1)/(z+1)$ is real only for real z. This is seen as follows:

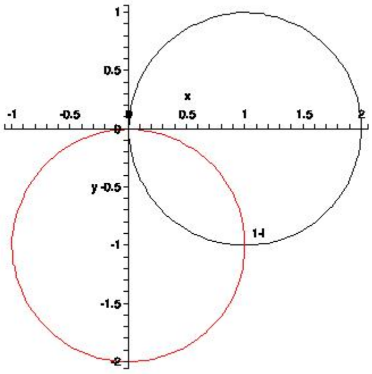
$$\frac{z-1}{z+1}\in\mathbb{R}\Leftrightarrow\frac{z-1}{z+1}=\frac{\bar{z}-1}{\bar{z}+1}\Leftrightarrow z\bar{z}+z-\bar{z}-1=z\bar{z}+\bar{z}-z-1\Leftrightarrow 2(z-\bar{z})=0\Leftrightarrow\Im(z)=0.$$

Moreover, it is a negative number exactly between -1 and 1. So on the complement of $[-1,1]$ the function takes nonnegative values. If we define the principal branch of the logarithm to be

$$\text{Log } w=\log|w|+i\text{Arg } w,\quad -\pi<\text{Arg } w<\pi,$$

then $F(z)$ as the composition of holomorphic maps is holomorphic and is the primitive of $1/(z^2-1)$ in a region that contains our contour. Consequently the integral is 0.

Remark: One can solve this problem with the residue theorem, which will be seen later in the course.



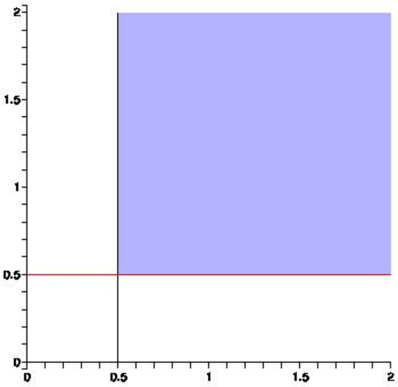


Figure 2: The region in problem 2 and its image by $w = z^{-1}$

4. Map conformally the region inside both circles $|z-1| < 1$ and $|z+i| < 1$ to the upper-half plane $\mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\}$.

   Hint: Use first the map $w = z^{-1}$ and at some later stage use $w = z^2$.

   Let $C_1 = \{z : |z-1| = 1\}$ and $C_2 = \{z : |z+i| < 1\}$. The map $w = T(z) = z^{-1}$ maps the two circles to two lines, as $T(0) = \infty$ and 0 belongs to both circles. The second common point of the two circles is $1-i$ (check this!). This is mapped to $T(1-i) = 1/2 + i/2$. The two circles are orthogonal at 0, as the tangent lines there are the real and the imaginary axis. This is also true at $1-i$ (the tangent lines are horizontal and vertical). So the images of the two circles are two lines meeting perpendicularly at $1/2 + i/2$. Since $T(2) = 1/2$, $T(C_1) = \{w : \Re w = 1/2\}$. Since $T(-2i) = i/2$, $T(C_2) = \{w : \Im w = 1/2\}$. Moreover, $T(1) = 1$, which lies to the right of $T(C_1)$. Using arguments similar to the previous homework, we deduce that the interior of C_1 is mapped to the half-plane on the right of the vertical line $\Re(w) = 1/2$. As $T(-i) = i$, which lie above the line $T(C_2)$, we can also deduce that the interior of C_2 is mapped into the half-plane above the line $\Im(w) = 1/2$. These imply that the region between the two circles is mapped to the set $\Omega = \{w : \Re w > 1/2, \Im w > 1/2\}$. Now translate Ω to the first quadrant by the map $S(w) = w - (1/2 + i/2)$. The first quadrant is mapped conformally to the upper half plane by the squaring function $H(z) = z^2$. The final map is the composition $H \circ S \circ T$.

5. Let C be the boundary of the triangle with vertices at the points $z = 0$, $z = 3i$ and $z = -4$, traversed anticlockwise. Without evaluating the integral show that

$$\left| \int_C (e^z - \bar{z})\, dz \right| \leq 60.$$

   The length of the triangle is $3 + 4 + 5 = 12$, as it is a right triangle with hypotenuse of length 5 and vertical sides of lengths 3 and 4. Moreover,

$$|e^z - \bar{z}| \leq |e^z| + |\bar{z}| = e^x + |z| \leq e^0 + 4 = 5.$$

Here we have used that the triangle lies in the second quadrant, where $x \leq 0$. Then

$$\left| \int_C (e^z - \bar{z})\, dz \right| \leq \text{length}(C) \max_{z \in C} |e^z - \bar{z}| \leq 12 \times 5 = 60.$$

6. $*$ (a) By considering the contour integral

$$\int_{|z|=1} \left(z + \frac{1}{z} \right)^{2n} \frac{dz}{z}$$

prove that

$$\int_0^{2\pi} \cos^{2n} t\, dt = 2\pi \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

We use the binomial theorem for $(z + 1/z)^n$ to get

$$\int_{|z|=1} \sum_{j=0}^{2n} \binom{2n}{j} z^j (z^{-1})^{2n-j} z^{-1} dz = \sum_{j=0}^{2n} \binom{2n}{j} \int_{|z|=1} z^{2j-2n-1} dz = 2\pi i \binom{2n}{n},$$

since $\int_{|z|=1} z^k dz = 2\pi i$ if $k = -1$ and 0 otherwise. Now we substitute $z = e^{it}$ to get (use $2\cos t = e^{it} + e^{-it}$)

$$\int_{|z|=1} \left(z + \frac{1}{z} \right)^{2n} \frac{dz}{z} = \int_0^{2\pi} (2\cos t)^{2n} \frac{ie^{it}}{e^{it}} dt = 2^{2n} i \int_0^{2\pi} \cos^{2n} t\, dt.$$

Comparing the two results we get

$$\int_0^{2\pi} \cos^{2n} t\, dt = \frac{2\pi}{2^{2n}} \binom{2n}{n} = \frac{2\pi}{2^{2n}} \frac{(2n)!}{n!n!} = \frac{2\pi 1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n-1) \cdot (2n)}{2^{2n} 1 \cdot 2 \cdots n \cdot 1 \cdot 2 \cdots n}.$$

We cancel the even numbers from the numerator with a factor of 2 and an integer from 1 to n. This gives

$$\int_0^{2\pi} \cos^{2n} t\, dt = \frac{2\pi \cdot 1 \cdot 3 \cdots (2n-1)}{2^n \cdot 1 \cdot 2 \cdots n}$$

and this gives the result by doubling every integer from 1 to n to get the even integers from 2 to $2n$ in the denominator.

(b) Prove that

$$\int_0^{\pi/2} \sin^{2n} t\, dt = \frac{\pi}{2} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

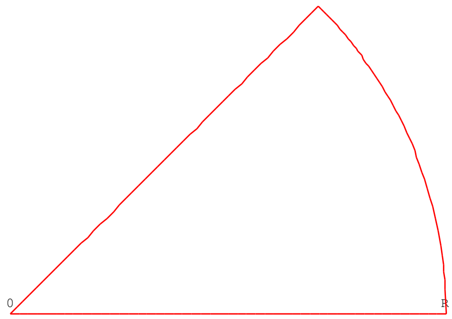


Figure 3: The contour for the Exercise 7

The substitution $t = x + \pi/2$ gives in (a) (use that $\cos t = \cos(x + \pi/2) = -\sin x$)

$$\int_0^{2\pi} \cos^{2n} t \, dt = \int_{-\pi/2}^{3\pi/2} \sin^{2n} x \, dx = \int_0^{2\pi} \sin^{2n} x \, dx$$

using the periodicity of $\sin x$. Now we remark that the values of $\sin^2 x$ on $[0, \pi/2]$ are the same on the other three intervals $[\pi/2, \pi]$, $[\pi, 3\pi/2]$ and $[3\pi/2, 2\pi]$, so the same is true for $\sin^{2n} x$. This means that the integral we want is $1/4$ the result from (a).

7. $*$ Evaluate the integrals

$$\int_0^{\infty} e^{-ax} \cos(bx) \, dx, \qquad \int_0^{\infty} e^{-ax} \sin(bx) \, dx, \quad a, b > 0$$

by integrating the holomorphic function e^{-Az}, $A = \sqrt{a^2 + b^2}$, over a sector with angle ω satisfying

$$\cos \omega = \frac{a}{A}.$$

Do not use integration by parts.

On the segment $\gamma_1 = [0, R]$ we set $z = x$, $dz = dx$ and integrate to get

$$\int_0^R e^{-Ax} \, dx = \left[\frac{e^{-Ax}}{-A} \right]_0^R = \frac{1}{A}(1 - e^{-AR}) \to \frac{1}{A} = \frac{1}{\sqrt{a^2 + b^2}}, \quad R \to \infty.$$

On the arc γ_R given by $\gamma(t) = Re^{it}$, $0 \le t \le \omega$, we have

$$\Re(z) \ge \Re(Re^{i\omega}) = R \cos \omega = Ra/A.$$

Moreover,

$$|e^{-Az}| = e^{-A\Re z} \leq e^{-ARa/A} = e^{-Ra}.$$

This gives

$$\left|\int_{\gamma_R} e^{-Az}\, dz\right| \leq \text{length } (\gamma_R) \max_{z\in\gamma_R} |e^{-Az}| \leq R\frac{\omega}{2\pi} e^{-Ra} \to 0. \quad R \to \infty.$$

We have used the fact that the function e^{-aR} is exponentially decaying, so multiplying by R does not affect it. If you want a formal proof, use L'Hôpital's rule to R/e^{aR}.

On the radial segment γ_2 from 0 to $Re^{i\omega}$ we have $z = te^{i\omega}$, $0 \leq t \leq R$, $dz = e^{i\omega}dt$ so that (notice that $b = A\sin\omega$)

$$\int_{\gamma_2} e^{-Az}\, dz = \int_0^R e^{-A(t\cos\omega + it\sin\omega)} e^{i\omega} dt = e^{i\omega} \int_0^R e^{-at} e^{-ibt}\, dt$$

$$= e^{i\omega} \int_0^R e^{-at}(\cos bt - i\sin bt)\, dt = e^{i\omega} \left(\int_0^R e^{-at}\cos(bt)\, dt - i\int_0^R e^{-at}\sin(bt)\, dt\right)$$

By Cauchy's Theorem we have

$$\int_{\gamma_1} e^{-Az}\, dz + \int_{\gamma_R} e^{-Az}\, dz - \int_{\gamma_2} e^{-Az}\, dz = 0.$$

We let $R \to \infty$ to deduce that

$$\frac{1}{\sqrt{a^2+b^2}} + 0 - e^{i\omega}\left(\int_0^\infty e^{-at}\cos(bt)\, dt - i\int_0^\infty e^{-at}\sin(bt)\, dt\right) = 0$$

$$\implies \int_0^\infty e^{-at}\cos(bt)\, dt - i\int_0^\infty e^{-at}\sin(bt)\, dt = e^{-i\omega}\frac{1}{\sqrt{a^2+b^2}} = \frac{\cos\omega - i\sin\omega}{\sqrt{a^2+b^2}}.$$

We make the real and imaginary parts equal to deduce that

$$\int_0^\infty e^{-at}\cos(bt)\, dt = \frac{\cos\omega}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2},$$

$$\int_0^\infty e^{-at}\sin(bt)\, dt = \frac{\sin\omega}{\sqrt{a^2+b^2}} = \frac{b}{a^2+b^2}.$$

Math 2101

Homework 6

Due: November 18, 2013

1. $*$ Let C be the boundary of the square with sides along the lines $x = \pm 2$ and $y = \pm 2$ traversed anticlockwise. Compute

$$\int_C \frac{\cos z}{z(z^2+8)}\, dz, \quad \int_C \frac{\cosh z}{z^3}\, dz, \quad \int_C \frac{1}{(z^2+1)^2}\, dz.$$

Do not use the residue theorem.

(a) We notice that the denominator vanishes at 0 and when $z^2 = -8 \Leftrightarrow z = \pm i\sqrt{8}$. As the last two points are outside the square, the function

$$f(z) = \frac{\cos z}{z^2+8}$$

is holomorphic on a domain containing the contour (four sides) and its interior (the square). We apply Cauchy's integral formula with $z_0 = 0$ to get

$$\frac{1}{8} = \frac{\cos 0}{0^2+8} = f(0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-0}\, dz = \frac{1}{2\pi i}\frac{\cos z}{z(z^2+8)}\, dz$$

this implies that

$$\int_C \frac{\cos z}{z(z^2+8)}\, dz = \frac{\pi i}{4}.$$

(b) We notice that 0 is inside the square. Cauchy's formula for $f(z)$ at $z_0 = 0$ for the second derivative gives:

$$f''(z_0) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3} dz = \frac{2}{2\pi i}\int_C \frac{\cosh z}{z^3} dz.$$

We compute

$$f(z) = \cosh z, \quad f'(z) = \sinh z, \quad f''(z) = \cosh z$$

so that $f''(0) = f''(z_0) = \cosh 0 = 1$. Therefore the given integral has the value $i\pi$.

(c) We use partial fractions, noticing that $z^2 + 1 = (z+i)(z-i)$. We set

$$\frac{1}{(z^2+1)^2} = \frac{A}{z+i} + \frac{B}{(z+i)^2} + \frac{C}{z-i} + \frac{D}{(z-i)^2}.$$

This gives

$$[A(z+i)+B](z-i)^2+[C(z-i)+D](z+i)^2=1.$$

We plug $z=i$ to get $D(2i)^2=1 \Longrightarrow D=-1/4$.

We plug $z=-i$ to get $B(-2i)^2=1 \Longrightarrow B=-1/4$.

We compare the coefficient of z^3 from both sides to get $A+C=0$.

We calculate the constant from both sides to get

$$(Ai+B)i^2+(-iC+D)i^2=1 \Leftrightarrow i(A-C)+B+D=-1 \Leftrightarrow i(A-C)=-1+1/4+1/4=-1/2.$$

So we get the system

$$A+C=0, \quad A-C=-1/(2i).$$

The solution is $A=-1/(4i)$ and $C=1/(4i)$. We now use Cauchy's integral formula to the constant function with derivative 0 to see that

$$\frac{1}{2\pi i}\int_C \frac{1}{(z^2+1)^2}\,dz = \frac{1}{2\pi i}\int_C \left(\frac{-1/(4i)}{z+i}+\frac{-1/4}{(z+i)^2}+\frac{1/(4i)}{z-i}+\frac{-1/4}{(z+i)^2}\right) dz$$

$$= \frac{1}{2\pi i}\left(-\frac{1}{4i}+0+\frac{1}{4i}+0\right)=0.$$

The answer is 0 for the given integral.

2. $*$ By evaluating the integral

$$\frac{1}{2\pi i}\int_C \frac{1}{(z-a)(z-a^{-1})}\,dz$$

around the unit circle C, prove that, if $0<a<1$, we have

$$\int_0^{2\pi} \frac{1}{1+a^2-2a\cos t}\,dt = \frac{2\pi}{1-a^2}.$$

You are not allowed to use the residue theorem.

Since a lies inside the unit circle, while $1/a$ does not, we apply Cauchy's formula to the holomorphic function

$$f(z)=\frac{1}{z-a^{-1}}$$

to get

$$f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz = \frac{1}{2\pi i}\int_C \frac{1}{(z-a)(z-a^{-1})}dz = \frac{1}{a-a^{-1}} = \frac{a}{a^2-1}.$$

On the other hand with the parametrisation $z=e^{it}$, we have $z+z^{-1}=2\cos t$ and $dz=ie^{it}dt=izdt$. We get

$$\frac{1}{2\pi i}\int_C \frac{1}{z^2-(a+a^{-1})z+1}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{iz\,dt}{z^2-(a+a^{-1})z+1} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{z+z^{-1}-(a+a^{-1})}dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\cos t-(a+a^{-1})}dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{a}{2a\cos t-(a^2+1)}dt = \frac{a}{a^2-1}.$$

We cancel a from both sides, multiply with -2π to get the desired result.

3. $*$ By considering the contour integral

$$\int_C \frac{e^{az}}{z}\,dz$$

around the unit circle C traversed anticlockwise, show that

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi.$$

We apply Cauchy's integral formula to $f(z)=e^{az}$ with $z_0=0$. We get

$$1 = e^{a0} = f(0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-0}\,dz = \frac{1}{2\pi i}\int_C \frac{e^{az}}{z}\,dz \Longrightarrow \int_C \frac{e^{az}}{z}\,dz = 2\pi i.$$

We now parametrise the circle C as $z=e^{i\theta}$, $-\pi\le\theta\le\pi$, so that $dz = ie^{i\theta}d\theta$ and

$$e^{az} = e^{a(\cos\theta+i\sin\theta)} = e^{a\cos\theta}e^{ia\sin\theta} = e^{a\cos\theta}(\cos(a\sin\theta)+i\sin(a\sin\theta)).$$

This gives

$$2\pi i = \int_C \frac{e^{az}}{z}\,dz = \int_{-\pi}^{\pi} e^{a\cos\theta}(\cos(a\sin\theta)+i\sin(a\sin\theta))\frac{ie^{i\theta}}{e^{i\theta}}d\theta$$

$$= i\int_{-\pi}^{\pi} e^{a\cos\theta}(\cos(a\sin\theta)+i\sin(a\sin\theta))d\theta = i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta - \int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)d\theta.$$

We take the imaginary part form the left and the right to get

$$2\pi = \int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = \int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta + \int_{-\pi}^{0} e^{a\cos\theta}\cos(a\sin\theta)d\theta. \quad (1)$$

In the second integral we make the change of variable $\theta=-t$ so that $d\theta=-dt$ and change the limits of integration accordingly to get:

$$\int_{-\pi}^{0} e^{a\cos\theta}\cos(a\sin\theta)d\theta = \int_{\pi}^{0} e^{a\cos(-t)}\cos(a\sin(-t))(-dt) = \int_0^{\pi} e^{a\cos t}\cos(-a\sin t)dt$$

$$= \int_0^{\pi} e^{a\cos t}\cos(a\sin t)\,dt$$

using the fact that cos is even and sin is odd function. The last integral is the same as the first integral in (1). Therefore, we have

$$2\pi = 2\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta$$

and this gives the result required.

4. Show that, when f is holomorphic on an open set U containing the simple closed contour C and its interior, and z_0 is not on C, then

$$\int_C \frac{f'(z)}{z-z_0}\,dz = \int_C \frac{f(z)}{(z-z_0)^2}\,dz.$$

We use Cauchy's integral formula for the first derivative and the function f to see that

$$\int_C \frac{f(z)}{(z-z_0)^2}\,dz = 2\pi i f'(z_0).$$

On the other hand we use Cauchy's integral formula for the function $g(z) = f'(z)$ to see that

$$f'(z_0) = g(z_0) = \frac{1}{2\pi i}\int_C \frac{g(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{z-z_0}\,dz.$$

The result becomes now obvious.

5. Let $f(z)$ be holomorphic on the unit disc and $f(0) = 1/2$. By working with

$$\frac{1}{2\pi i}\int_{|z|=1}\left[2 \pm (z+\frac{1}{z})\right] f(z)\frac{dz}{z}$$

prove that

$$\frac{2}{\pi}\int_0^{2\pi} f(e^{it})\cos^2\frac{t}{2}\,dt = 1 + f'(0), \quad \frac{2}{\pi}\int_0^{2\pi} f(e^{it})\sin^2\frac{t}{2}\,dt = 1 - f'(0).$$

We have $2\sin^2(t/2) = 1-\cos t$ and $2\cos^2(t/2) = 1+\cos t$. Parameterizing the circle as $z = e^{it}$ we get

$$\frac{1}{2\pi i}\int_{|z|=1}\left[2 \pm (z+\frac{1}{z})\right] f(z)\frac{dz}{z} = \frac{1}{2\pi i}\int_0^{2\pi}(2\pm 2\cos t)f(e^{it})i\,dt$$

$$= \frac{1}{\pi}\int_0^{2\pi}(1\pm\cos t)f(e^{it})dt = \frac{2}{\pi}\int_0^{2\pi}\left\{\begin{array}{c}\cos^2(t/2)\\ \sin^2(t/2)\end{array}\right\} f(e^{it})dt.$$

This is how we get the two integrals on the left-hand side of the result. For the right-hand sides we use the Cauchy Integral formulas:

$$\frac{1}{2\pi i}\int_{|z|=1} f(z)\frac{dz}{z} = f(0) = 1/2, \quad \frac{1}{2\pi i}\int_{|z|=1|}\frac{f(z)}{z^2}dz = f'(0),$$

while Cauchy's theorem gives directly

$$\frac{1}{2\pi i}\int_{|z|=1} f(z)\,dz = 0.$$

As a result

$$\frac{1}{2\pi i}\int_{|z|=1}\left[2 \pm (z+\frac{1}{z})\right] f(z)\frac{dz}{z} = \frac{1}{2\pi i}\int_{|z|=1}\left(2\frac{f(z)}{z} \pm (f(z)+\frac{f(z)}{z^2})\right)dz = 1 \pm f'(0).$$

6. $*$ Let C be a circle enclosing the distinct points $z_1, z_2, \ldots, z_n$. Let

$$p(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$$

be a polynomial of degree n with roots at these points. Let $f(z)$ be holomorphic in a disc that includes C. Show that

$$P(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{p(w)} \frac{p(w) - p(z)}{w - z} dw$$

is a polynomial of degree $n - 1$, with the property

$$P(z_i) = f(z_i), \quad i = 1, 2, \ldots n.$$

We plug z_i to get

$$P(z_i) = \frac{1}{2\pi i} \int_C \frac{f(w)}{p(w)} \frac{p(w) - p(z_i)}{w - z_i} dw = \frac{1}{2\pi i} \int_C \frac{f(w)}{p(w)} \frac{p(w)}{w - z_i} dw = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_i} dw = f(z_i)$$

by Cauchy's formula. It suffices to prove that it is a polynomial of degree $\leq n - 1$. We expand $p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0$ and get

$$\frac{p(w) - p(z)}{w - z} = \frac{w^n - z^n + a_{n-1}(w^{n-1} - z^{n-1}) + \cdots + a_1(w - z)}{w - z}$$

$$= w^{n-1} + w^{n-2}z + \cdots + wz^{n-2} + z^{n-1} + a_{n-1}(w^{n-2} + w^{n-3}z + \cdots + wz^{n-3} + z^{n-2}) + \cdots + a_1,$$

where we used that $A^N - B^N = (A - B)(A^{N-1} + A^{N-2}B + \cdots + AB^{N-1} + B^N)$. This is clearly a polynomial in z of degree $n - 1$. When we plug it into our integral in the variable w we get by linearity a polynomial of degree $\leq n - 1$ in z with coefficients the values of certain contour integrals in w.

The following exercise is more theoretical.

7. Let f is holomorphic on an open set U containing the closed disk D with boundary the circle C. Show the Cauchy formulae for higher derivatives, which was proved in class for $n = 1$:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz, \quad \forall n \in \mathbb{N},$$

where C is traversed anticlockwise and z_0 is inside C.

Hint: Use induction. Imitate the proof from the class. It helps to define $A = z - z_0$, $B = z - z_0 - h$ and recall the identity

$$A^k - B^k = (A - B)(A^{k-1} + A^{k-2}B + A^{k-3}B^2 + \cdots + AB^{k-2} + B^{k-1}).$$

For $n = 1$ we proved it in class. Assume the formula is true for k, i.e.

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{k+1}} \, dz, \quad \forall z_0 \in U,$$

we need to prove it for $n = k+1$, i.e.

$$f^{(k+1)}(z_0) = \frac{(k+1)!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{k+2}}\, dz. \tag{2}$$

We look at the difference quotient and for the two terms in the numerator we use the inductive hypothesis:

$$\frac{f^{(k)}(z_0+h) - f^{(k)}(z_0)}{h} = \frac{k!}{2\pi i h} \int_C \left(\frac{f(z)}{(z-(z_0+h))^{k+1}} - \frac{f(z)}{(z-z_0)^k} \right) dz$$

To prove that the function $f^{(k)}(z)$ is differentiable at z_0 with derivative the right hand side of (2), it is enough to show that

$$\lim_{h\to 0} \left(\frac{k!}{2\pi i h} \int_C \left(\frac{f(z)}{(z-(z_0+h))^{k+1}} - \frac{f(z)}{(z-z_0)^k} \right) dz - \frac{(k+1)!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{k+2}}\, dz \right) = 0.$$

With the notation $A = z - z_0$, $B = z - z_0 - h$, it suffices to prove (first cancel $k!/(2\pi i)$)

$$\lim_{h\to 0} \int_C f(z) \left((A-B)^{-1} \left(B^{-(k+1)} - A^{-(k+1)} \right) - (k+1) A^{-(k+2)} \right) dz = 0.$$

We now do the algebra with the expression involving A and B:

$$\begin{aligned} \frac{1}{A-B} \left(B^{-(k+1)} - A^{-(k+1)} \right) - (k+1) A^{-(k+2)} &= \frac{1}{A-B} \frac{A^{k+1} - B^{k+1}}{A^{k+1} B^{k+1}} - \frac{k+1}{A^{k+2}} \\ &= \frac{(A^k + A^{k-1}B + \cdots + AB^{k-1} + B^k)A}{A^{k+2}B^{k+1}} - \frac{(k+1)B^{k+1}}{A^{k+2}B^{k+1}}. \end{aligned}$$

We split the $(k+1)B^{k+1}$ into $k+1$ terms of size B^{k+1}. So we have to deal with terms of the type $(j = 0, \ldots, k)$

$$A^{k-j} B^j A - B^{k+1} = B^j (A^{k-j+1} - B^{k+1-j}).$$

We easily see that

$$\lim_{h\to 0} (A^{k-j+1} - B^{k+1-j}) = \lim_{h\to 0} ((z-z_0)^{k-j+1} - (z-z_0-h)^{k-j+1}) = 0,$$

as the exponents are always positive. On the other hand the denominators give

$$\lim_{h\to 0} A^{k+2} B^{k+1} = A^{2k+3} \neq 0$$

as z_0 is inside the curve C, so at a positive distance $d = \min_{z\in C} |z - z_0|$ from C. We recall that f is bounded on C by $\max_{z\in C} |f(z)|$. We are left with $k+1$ integrals to estimate

$$\left| \int_C f(z) \frac{B^j (A^{k-j+1} - B^{k+1-j})}{A^{k+2} B^{k+1}}\, dz \right| \le L(C) \max_{z\in C} |f(z)| \max_{z\in C} \frac{|(z-z_0)^{k-j+1} - (z-z_0-h)^{k-j+1}|}{|(z-z_0)^{k+2} (z-z_0-h)^{k+1-j}|},$$

which tends to 0 as $h \to 0$.

Math 2101

Homework 7

Due: November 27, 2013

1. Let $f(z)$ be holomorphic in the region $|z| \leq R$ with power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let the partial sum of the series be defined as

$$s_N(z) = \sum_{n=0}^{N} a_n z^n.$$

Show that for $|z| < R$ we have

$$s_N(z) = \frac{1}{2\pi i} \int_{|w|=R} f(w) \frac{w^{N+1} - z^{N+1}}{w - z} \frac{dw}{w^{N+1}}.$$

We use the identity $A^N - B^N = (A - B)(A^{N-1} + A^{N-2}B + \cdots + AB^{N-1} + B^N)$. We, therefore, get

$$\frac{1}{2\pi i} \int_{|w|=R} f(w) \frac{w^{N+1} - z^{N+1}}{w - z} \frac{dw}{w^{N+1}} = \frac{1}{2\pi i} \int_{|w|=R} f(w)(w^N + w^{N-1}z + \cdots + wz^{N-1} + z^N) \frac{dw}{w^{N+1}}$$

$$= \frac{1}{2\pi i} \int_{|w|=R} f(w)(w^{-1} + w^{-2}z + w^{-3}z^2 \cdots + w^{-N}z^{N-1} + w^{-N-1}z^N) dw$$

$$= f(0) + f'(0)z + \frac{f''(0)}{2} z^2 + \cdots + \frac{f^{(N-1)}(0)}{(N-1)!} z^{N-1} + \frac{f^{(N)}(0)}{N!} z^N$$

by Cauchy's formula for the derivatives. Since $a_n = f^{(n)}(0)/n!$ we get that the result is exactly $s_N(z)$.

2. $*$ Let f be entire.

(a) Show that, if e^f is bounded, then f is constant.

Since e^f is entire, as composition of e^w and f, and bounded, by assumption, we can apply Liouville's theorem and deduce that e^f is constant, say $= k$. Now

$$e^{f(z)} = k \Longrightarrow f(z) = \log k = \log |k| + i \arg k.$$

The trouble is that the argument of k is not unique. So we cannot immediately see that f is constant. All the arguments of k differ by integer multiples of $2\pi i$. Assume that we can find z_1 and z_2 different complex numbers with

$$f(z_1) = \log |k| + arg(k), \quad f(z_2) = \log |k| + arg(k) + 2\pi i n, \quad n \neq 0, n \in \mathbb{Z}.$$

Here $arg(k)$ denotes a fixed argument of k. As f is continuous, so is its imaginary part $\Im(f)$. We join z_1 and z_2 with a line segment $t \to tz_1 + (1-t)z_2$, $t \in [0,1]$. The function

$$\phi : [0.1] \to \mathbb{R}, \quad \phi(t) = \Im(f(tz_1 + (1-t)z_2))$$

is continuous so, by the intermediate value theorem, it assumes all values between $arg(k)$ and $arg(k) + 2\pi n$, e.g $arg(k) + \pi n$. On the other hand, $\Im f(z)$ is always an argument of k, and $arg(k) + \pi n$ is not. Said differently, $(\Im(f) - arg(k))/(2\pi)$ takes only integer values, while the intermediate value theorem will imply it should take all values between 0 and n. We can conclude that such z_1 and z_2 do not exist and for all $z \in \mathbb{C}$ $\Im f(z)$ is the same fixed argument of k. Therefore, f is constant.

(b) Assume that $\Im(f)$ is bounded below. Show that f is a constant function.

Assume that $\Im(f(z)) \geq m$ for all $z \in \mathbb{C}$. Consider the entire function $g(z) = if(z)$. We have

$$|e^{g(z)}| = |e^{if(z)}| = e^{-\Im(f(z))} \leq e^{-m}.$$

By (a) $g(z) = if(z)$ is a constant, which implies that f is a constant.

3. For each of the following functions determine the isolated singularities and their nature. For the poles, find the order of the pole, the principal part and residue at the pole.

$$\text{(a)}\ \frac{z^2}{1+z}, \quad \text{(b)}\ \frac{\sin z}{z}, \quad \text{(c)}\ \frac{\cos z}{z} \quad \text{(d)}\ \tanh z.$$

(a) This is a rational function and is holomorphic whenever the denominator is not zero. The only singularity if -1 and it is isolated, as $f(z) = z^2/(1+z)$ is holomorphic in $\mathbb{C} \setminus \{-1\}$. The singularity is a pole as $1/f(z) = (1+z)/z^2$ has a simple zero at -1. This follows from the fact that $1/z^2$ is holomorphic and nonzero in $\mathbb{C} \setminus \{0\}$. We conclude that -1 is a simple pole for $f(z)$. To determine the principal part we write

$$f(z) = \frac{z^2}{1+z} = \frac{z^2 - 1 + 1}{1+z} = \frac{(z+1)(z-1)}{z+1} + \frac{1}{z+1} = z - 1 + \frac{1}{z+1}.$$

Since $z - 1$ is holomorphic, we see that the principal part is $1/(z+1$ and the residue is 1. Alternatively, we can calculate the residue as

$$\text{res}(f, -1) = \lim_{z \to -1} (z+1) f(z) = \lim_{z \to -1} z^2 = 1.$$

Since the pole is simple, there is only one term with negative exponent in the Laurent series of $f(z)$ and it is $1/(z+1)$.

(b) Since $f(z) = (\sin z)/z$ is holomorphic with the exception of 0, we see that 0 is an isolated singularity. By the Taylor series of $\sin z$ we see that

$$f(z) = \frac{z - z^3/6 + z^5/5! - \cdots}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

and this is a convergent power series with infinite radius of convergence (the coefficients have not changed from the Taylor series of $\sin z$, it is only the exponents that are one smaller). This power series is holomorphic on $\mathbb{C}$. Moreover, the value of the power series at 0 is 1. Therefore, if we define $f(0) = 1$, we see that $f(z)$ is holomorphic on $\mathbb{C}$. The singularity at 0 is a removable singularity.

(c) Since $f(z) = (\cos z)/z$ is holomorphic with the exception of 0, we see that 0 is an isolated singularity. Since $1/f(z) = z/\cos(z)$ is holomorphic for $z \in D(0, \pi/2)$ (the only trouble are the zeros of $\cos(z)$ in the denominator), and $1/\cos(z)$ is nonzero in the same region, we see that $1/f$ has a simple zero at 0. So f has a simple pole at zero. We determine the principar part: By the Taylor series of $\cos z$ we see that

$$f(z) = \frac{1 - z^2/2 + z^4/4! - z^6/6! + \cdots}{z} = \frac{1}{z} + \left(-\frac{z}{2} + \frac{z^3}{4!} - \frac{z^5}{6!} + \cdots\right).$$

The power series in parentheses is holomorphic on $\mathbb{C}$. This is so, because with the exception of the first term that is missing, the coefficients are the same as the series for $\cos(z)$ but the exponents are just one less. We see that the singularity is a simple pole, the principal part is $1/z$ and the residue is 1.

(d) We write

$$f(z) = \tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1}.$$

This is a quotient of holomorphic functions, so it is holomorphic whenever the denominator is nonzero. We solve

$$e^{2z} + 1 = 0 \Leftrightarrow e^{2z} = -1 = e^{\pi i} \Leftrightarrow 2z = \pi i + 2\pi i n, \quad n \in \mathbb{Z} \Leftrightarrow z = \pi i(1/2 + n).$$

The function is holomorphic in open discs centered at these point with radii $\pi/2$. We deduce that these are isolated singularities. We will show that these points $r_n = \pi i(1/2 + n)$ are simple poles. Consider

$$\frac{1}{f(z)} = \frac{e^{2z} + 1}{e^{2z} - 1}.$$

We will show that it has simple zeros at r_n. We expand $g(z) = e^{2z} + 1$ into power series centered at r_n, using that $g^{(n)}(z) = 2^n e^{2z}$, and that $e^{2r_n} = -1$ to get

$$e^{2z} + 1 = -2(z - r_n) - \frac{2^2}{2!}(z - r_n)^2 - \frac{2^3}{3!}(z - r_n)^3 - \cdots.$$

We also notice that $e^{2z} - 1 \to -2$ as $z \to r_n$, so $e^{2z} - 1$ is holomorphic and non zero in a small disc around r_n (inertia principle). We get

$$\frac{1}{f(z)} = (z - r_n)\frac{-2 - (2^2/2!)(z - r_n) - (2^3/3!)(z - r_n)^2 - \cdots}{e^{2z} - 1}.$$

The numerator of the fraction is a power series representing an entire function that does not vanish at r_n, as it has the value -2 there. The fraction represents a holomorphic function in the small disc $D(r_n, \delta)$ which does not vanish on it. So $f(z)$ has a simple pole at r_n. To find the principal part it suffices to compute the residue:

$$\operatorname{res}(f, r_n) = \lim_{z\to r_n} f(z) = \lim_{z\to r_n} \frac{z - r_n}{e^{2z}+1}(e^{2z}-1) = \lim_{z\to r_n} \frac{z-r_n}{e^{2z}-e^{2r_n}}(e^{2z}-1)$$

$$= \frac{1}{2e^{2r_n}}(-2) = \frac{1}{-2}(-2) = 1$$

using the definition of the derivative of the function e^{2z} at r_n. The principal part is $1/(z - r_n)$ at the pole r_n.

4. What is the value of the integral

$$\int_C \frac{1}{z^2+1}\, dz,$$

where C is (i) the circle $|z| = 2$ traversed anticlockwise, (ii) the circle $|z - i| = 1$ traversed anticlockwise?

We have

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}.$$

For (ii) we notice that $f(z) = 1/(z + i)$ is holomorphic on and inside the circle $|z - i| = 1$. So Cauchy's integral formula gives

$$\int_C \frac{1}{(z-i)(z+i)}\, dz = 2\pi i f(i) = 2\pi i \frac{1}{i+i} = \pi.$$

For (i) we use partial fractions:

$$\frac{1}{z^2+1} = \frac{1/(2i)}{z-i} + \frac{-1/(2i)}{z+i}.$$

We set $f_1(z) = 1/(2i)$. Then

$$\int_C \frac{1}{(z-i)(z+i)}\, dz = \int_C \frac{1/(2i)}{z-i}\, dz - \int_C \frac{1/(2i)}{z+i}\, dz = 2\pi i f_1(i) - 2\pi i f_1(-i) = 0,$$

as f_1 is constant. We have used Cauchy's integral formula twice.

Alternative method: Use Cauchy's residue theorem. With $f(z) = 1/(z^2 + 1)$ we have residues at $\pm i$. Therefore,

$$\text{res }(f, \pm i) = \lim_{z\to\pm i}(z \mp i) f(z) = \lim_{z\to\pm i} \frac{1}{z \pm i} = \frac{1}{\pm 2i}.$$

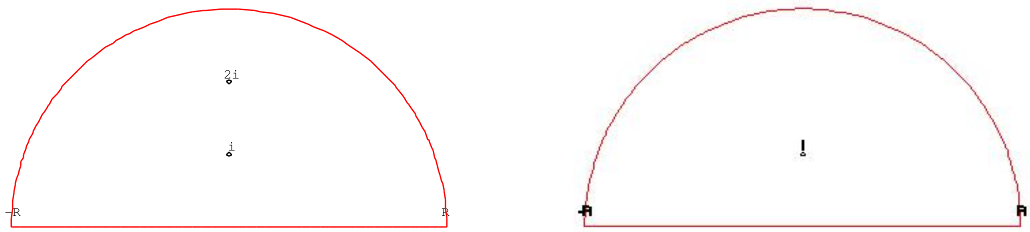


Figure 1: Contours for problem 5(a) and 5(b)

By the residue theorem for (i)

$$\int_C f(z)\,dz = 2\pi i(\text{res }(f, i) + \text{res }(f, -i)) = 0.$$

By the residue theorem for (ii)

$$\int_C f(z)\,dz = 2\pi i(\text{res }(f, i)2\pi i\frac{1}{2i} = \pi.$$

5. $*$ Explain why the following integrals have the given value using the residue theorem. Complete explanations are required.

$$\text{(a)} \int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{6}. \qquad \text{(b)} \int_0^\infty \frac{\cos(ax)}{x^2+1}\,dx = \frac{\pi}{2}e^{-a}, \quad a \geq 0.$$

(a) We consider

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$$

and the contour γ in Figure 1. By Cauchy's residue theorem

$$\int_\gamma f(z)\,dz = 2\pi i \sum_j \text{res }(f, z_j),$$

where z_j are the poles of $f(z)$ inside γ. The poles of f are at the zeros of the denominator $\pm i$ and $\pm 2i$. Only i and $2i$ are inside γ and only when $R > 2$. Since the zeros are all simple, the poles are simple. We calculate the residues.

$$\text{res }(f,i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}(z-i)\frac{z^2}{(z-i)(z+i)(z^2+4)} = \frac{i^2}{2i(i^2+4)} = \frac{-1}{2i\cdot 3} = -\frac{1}{6i}.$$

$$\text{res }(f,2i) = \lim_{z\to 2i}(z-2i)f(z) = \lim_{z\to 2i}(z-2i)\frac{z^2}{(z-2i)(z+2i)(z^2+1)} = \frac{(2i)^2}{4i((2i)^2+1)} = \frac{-4}{4i\cdot(-3)} = \frac{1}{3i}.$$

Therefore,

$$\int_\gamma f(z)\,dz = 2\pi i\left(-\frac{1}{6i}+\frac{1}{3i}\right) = 2\pi i\frac{1}{6i} = \frac{\pi}{3}.$$

The contour γ can be split in two parts: the horizontal segment $[-R,R]$ traversed from left to right and γ_R the semicircle traversed anticlockwise. On $[-R,R]$ we have the parametrisation $z = x$, $-R \le x \le R$, which gives $dz = dx$ and

$$\int_{[-R,R]} f(z)\,dz = \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = 2\int_0^R \frac{x^2}{(x^2+1)(x^2+4)}\,dx$$

as the integrand is an even function. If we show that

$$\lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz = 0$$

then

$$\lim_{R\to\infty}\left(\int_{[-R,R]} f(z)\,dz + \int_{\gamma_R} f(z)\,dz\right) = \frac{\pi}{3}$$

$$\implies \lim_{R\to\infty} 2\int_0^R \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{3}$$

$$\implies \int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{6}.$$

On γ_R we have $z = Re^{it}$, $|z^2+1| \ge |z|^2 - 1 = R^2 - 1$, $|z^2+4| \ge |z|^2 - 4 = R^2 - 4$ and, therefore,

$$\left|\int_{\gamma_R} f(z)\,dz\right| \le \text{length }(\gamma_R)\max_{z\in\gamma_R}|f(z)| \le \pi R\frac{R^2}{(R^2-1)(R^2-4)} \to 0$$

as $R \to \infty$, since the numerator is R^3 while there are four powers of R in the denominator.

(b) We use the function $f(z) = e^{iaz}/(1+z^2)$ inside the contour in Figure 1. The only pole is at i with residue

$$\text{Res}(f,i) = \lim_{z\to i}(z-i)\frac{e^{iaz}}{(z-i)(z+i)} = \frac{e^{iia}}{2i} = \frac{e^{-a}}{2i}.$$

We notice that

$$\int_{-R}^{R} \frac{e^{iaz}}{z^2+1}dz = \int_{0}^{R} \frac{e^{iax}}{x^2+1}dx + \int_{-R}^{0} \frac{e^{iax}}{x^2+1}dx = \int_{0}^{R} \frac{e^{iax}}{x^2+1}dx + \int_{0}^{R} \frac{e^{-iax}}{x^2+1}dx$$
$$= \int_{0}^{R} \frac{e^{iax}+e^{-iax}}{x^2+1}dx = 2\int_{0}^{R} \frac{\cos(ax)}{x^2+1}dx,$$

with a change on variable $x = -y$ on $[-R, 0]$. The semicircle is denoted by γ_R. On it $|e^{iaz}| = |e^{iaRe^{i\theta}}| = e^{-aR\sin\theta} \leq 1$, because $\theta \in [0, \pi]$. This gives

$$\left|\int_{\gamma_R} \frac{e^{iaz}}{z^2+1}dz\right| \leq \frac{\pi R}{R^2-1} \to 0, \quad R \to \infty,$$

using the triangle inequality in the denominator ($|z^2+1| \geq |z|^2 - 1 = R^2 - 1$). The result is that

$$2\int_{0}^{\infty} \frac{\cos(ax)}{x^2+1}dx = 2\pi i e^{-a}/(2i) = \pi e^{-a} \Longrightarrow \int_{0}^{\infty} \frac{\cos(ax)}{x^2+1}dx = \frac{\pi e^{-a}}{2}.$$

6. $*$ Explain why the following integrals have the given value using the residue theorem. Complete explanations are required.

$$\text{(a)} \int_{0}^{\infty} \frac{1}{x^6+1}\,dx = \frac{\pi}{3}, \quad \text{(b)} \int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+4)(x^2+16)}\,dx = \frac{\pi}{12}\left(\frac{e^{-2}}{2} - \frac{e^{-4}}{4}\right).$$

(a) We set

$$f(z) = \frac{1}{z^6+1},$$

which is holomorphic in $\mathbb{C}$ with the exceptions of the roots of the denominator. These are the sixth roots of -1. We compute them use De Moivre's theorem

$$z^6 = -1 = e^{\pi i} \Leftrightarrow z = e^{(\pi i + 2\pi i k)/6}, \quad k = 0, 1, \ldots, 5.$$

Rather than take the semicircular contour, we choose a sector bounded by the positive real axis and a ray $z = xe^{i\theta}$ for some appropriate θ. Since we have z^6+1 in the denominator, we can choose the angle θ so that on the ray z^6+1 is real and equal to x^6+1, if we set $e^{6i\theta} = 1$. This allows us to take $\theta = \pi/3$. With this choice of angle only one of the poles of $f(z)$ are inside the sector, the one corresponding to $z = e^{\pi i/6}$. We compute

$$\text{Res}(f, e^{\pi i/6}) = \lim_{z\to e^{\pi i/6}} (z - e^{\pi i/6})\frac{1}{z^6+1} = \lim_{z\to e^{\pi i/6}} \frac{1}{6z^5} = \frac{1}{6e^{5\pi i/6}}$$

by L'Hôpital's rule. We get on the horizontal segment $[0, R]$ that $z(x) = x$, on the segment from 0 to $Re^{\pi i/3}$ that $z(x) = xe^{\pi i/3}, 0 \leq x \leq R$. The part of the contour on a circle of radius R is denoted by γ_R. These give:

$$\int_{0}^{R} \frac{dx}{x^6+1} + \int_{\gamma_R} \frac{dz}{z^6+1} - \int_{0}^{R} \frac{e^{\pi i/3}dx}{x^6e^{2\pi i}+1} = 2\pi i \text{Res}(f, e^{\pi i/6}).$$

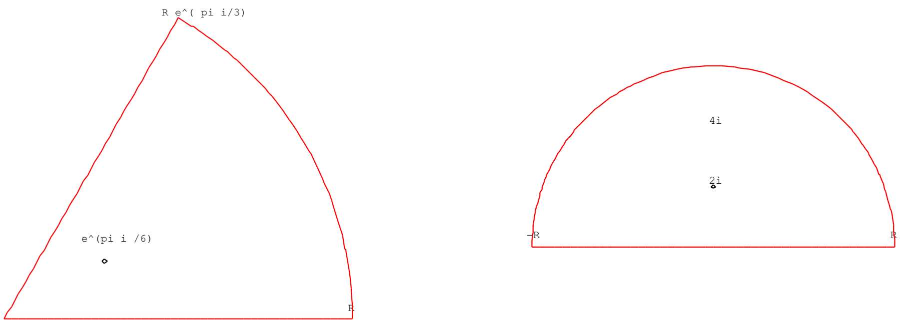


Figure 2: Contours for problem 6(a) and 6(b)

Notice that on the slanted segment $z^6 = (xe^{\pi i/3})^6 = x^6 e^{2\pi i} = x^6$. So we get

$$\int_0^R \frac{dx}{x^6+1} + \int_{\gamma_R} \frac{dz}{z^6+1} - e^{\pi i/3} \int_0^R \frac{dx}{x^6+1} = \frac{2\pi i}{6e^{5\pi i/6}}.$$

The integral over γ_R tends to 0 as $R \to \infty$. This is because on γ_R for $R > 1$ we have

$$\left| \frac{1}{z^6+1} \right| \leq \frac{1}{R^6-1}$$

by the triangle inequality ($|z^6+1| \geq |z|^6 - 1 = R^6 - 1$). On the other hand the length of the arc is $2\pi R/6$. This gives

$$\left| \int_{\gamma_R} \frac{dz}{z^6+1} \right| \leq \frac{2\pi R/6}{R^6-1} \to 0, \quad R \to \infty.$$

The result is

$$\int_0^\infty \frac{dx}{x^6+1} - e^{\pi i/3} \int_0^\infty \frac{dx}{x^6+1} = \frac{2\pi i}{6e^{5\pi i/6}}$$

$$\implies \int_0^\infty \frac{dx}{x^6+1} = \frac{2\pi i}{6e^{5\pi i/6}(1-e^{\pi i/3})} = \frac{2\pi i}{6(e^{5\pi i/6} - e^{7\pi i/6})}$$

$$= \frac{2\pi i}{6(e^{5\pi i/6} - e^{-5\pi i/6})} = \frac{2\pi i}{6 \cdot 2i \sin(5\pi/6)} = \frac{\pi}{6 \cdot 1/2} = \frac{\pi}{3}.$$

(b) We consider the semicircular contour. We set

$$f(z) = \frac{e^{iz}}{(z^2+4)(z^2+16)}.$$

We factor the denominator as $(z+2i)(z-2i)(z+4i)(z-4i)$. The function f is holomorphic apart from four simple poles at the roots of the denominator $\pm 2i$, $\pm 4i$. Only the poles $2i$ and $4i$ lie inside the contour and for $R > 4$ only. We calculate the residues at these points.

$$\begin{aligned}
\operatorname{res}(f, 2i) &= \lim_{z \to 2i} (z - 2i) f(z) = \lim_{z \to 2i} \frac{e^{iz}}{(z+2i)(z^2+16)} = \frac{e^{2i^2}}{(2i+2i)((2i)^2+16)} \\
&= \frac{e^{-2}}{4i(-4+16)} = \frac{e^{-2}}{48i}. \\
\operatorname{res}(f, 4i) &= \lim_{z \to 4i} (z - 4i) f(z) = \lim_{z \to 4i} \frac{e^{iz}}{(z+4i)(z^2+4)} = \frac{e^{4i^2}}{(4i+4i)((4i)^2+4)} \\
&= \frac{e^{-4}}{8i(-16+4)} = \frac{e^{-4}}{-96i}.
\end{aligned}$$

On the segment $[-R, R]$ the contribution is

$$\int_{-R}^{R} \frac{e^{ix}}{(x^2+4)(x^2+16)} dx = \int_{-R}^{R} \frac{\cos x}{(x^2+4)(x^2+16)} dx + i \int_{-R}^{R} \frac{\sin x}{(x^2+4)(x^2+16)} dx.$$

The contribution on the semicircle γ_R can be bounded as

$$\left| \int_{\gamma_R} f(z)\, dz \right| \le \pi R \max_{z \in \gamma_R} |f(z)| \le \pi R \frac{1}{(R^2-4)(R^2-16)} \to 0, \quad R \to \infty.$$

Here we have used that on γ_R we have

$$|z^2+4| \ge |z^2| - 4 = R^2 - 4, \quad |z^2+16| \ge |z^2| - 16 = R^2 - 16,$$
$$|e^{iz}| = e^{\Re(iz)} = e^{-\Im(z)} \le e^0 = 1.$$

The residue theorem gives

$$\int_{-R}^{R} \frac{e^{ix}}{(x^2+4)(x^2+16)} dx + \int_{\gamma_R} f(z)\, dz = 2\pi i \left(\frac{e^{-2}}{48i} + \frac{e^{-4}}{-96i} \right) = \frac{\pi}{12} \left(\frac{e^{-2}}{2} - \frac{e^{-4}}{4} \right).$$

We arrive at the conclusion by letting $R \to \infty$ and taking the real part in this equation.

7. (a) Show that on the interval $(0, \pi/2]$ the function $\sin u / u$ is decreasing. Deduce that

$$\sin u \ge \frac{2u}{\pi}, \quad u \in [0, \pi/2].$$

With $g(u) = \sin u / u$ for $u \ne 0$ we calculate the derivative

$$g'(u) = \frac{\cos u \cdot u - \sin u}{u^2}.$$

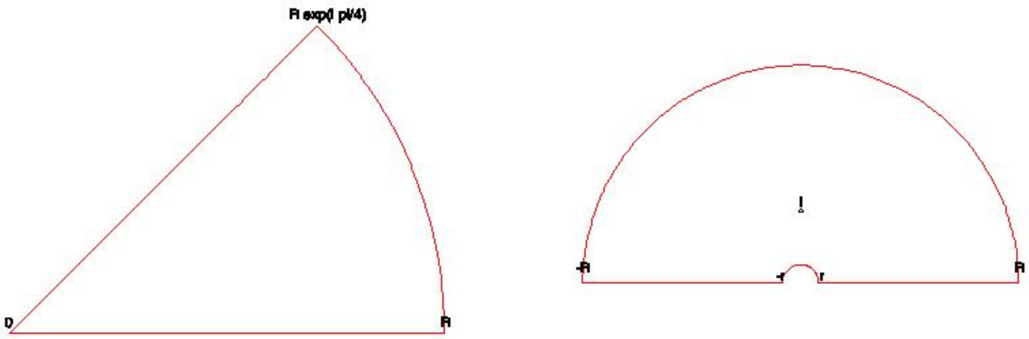


Figure 3: The contour for the Fresnel integrals and for problem 8

To show that g is decreasing, it suffices to show that $g'(u) \leq 0$. This is equivalent to

$$u \cos u - \sin u \leq 0 \Leftrightarrow u \cos u \leq \sin u \Leftrightarrow u \leq \tan u$$

for $u \in [0, \pi/2]$. This is well-known and can be proved using the mean value theorem.

$$\frac{\tan u - \tan 0}{u - 0} = \frac{1}{\cos^2 \xi}$$

for some $\xi \in (0, u)$. Since $\cos^2 \xi \leq 1$, we get

$$\frac{\tan u}{u} \geq 1 \Leftrightarrow \tan u \geq u.$$

(b) Use contour integration to show the value of the (Fresnel) integrals

$$\int_0^\infty \cos(x^2)\, dx = \int_0^\infty \sin(x^2)\, dx = \sqrt{2\pi}/4.$$

Hint: Consider the function $f(z) = e^{iz^2}$ and a contour that includes the line segment from 0 to $Re^{i\pi/4}$ and the circular arc from R to $Re^{i\pi/4}$.

Let the contour be $\gamma = \gamma_1 + \gamma_R - \gamma_2$, where $\gamma_1(t) = t + i0$, $0 \leq t \leq R$, $\gamma_R(\theta) = Re^{i\theta}$, $0 \leq \theta \leq \pi/4$, and, finally, $\gamma_2(t) = te^{i\pi/4}$, $0 \leq t \leq R$. By Cauchy's theorem

$$0 = \int_\gamma e^{iz^2} dz = \int_{\gamma_1} e^{iz^2} dz + \int_{\gamma_R} e^{iz^2} dz - \int_{\gamma_2} e^{iz^2} dz.$$

On γ_1 we have

$$\int_{\gamma_1} e^{iz^2} dz = \int_0^R e^{ix^2} dx = \int_0^R (\cos(x^2)+i\sin(x^2))dx = \int_0^R \cos(x^2)dx + i\int_0^R \sin(x^2)dx.$$

The last integral tends to

$$\int_0^\infty \cos(x^2)\,dx + i\int_0^\infty \sin(x^2)\,dx, \quad R \to \infty.$$

On γ_2 we have

$$\int_{\gamma_2} e^{iz^2} dz = \int_0^R e^{i(te^{i\pi/4})^2} e^{i\pi/4} dt = \int_0^R e^{it^2 i} e^{i\pi/4} dt = e^{i\pi/4}\int_0^R e^{-t^2}\,dt$$

$$\to e^{i\pi/4}\int_0^\infty e^{-t^2} dt = \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\frac{\sqrt{\pi}}{2}, \quad R \to \infty.$$

The result for the two Fresnel integrals will follow, if we show that the contour integral over γ_R tends to 0, $R \to \infty$. We have

$$\int_{\gamma_R} e^{iz^2} dz = \int_0^{\pi/4} e^{i(Re^{i\theta})^2} Rie^{i\theta}\,d\theta = \int_0^{\pi/4} e^{iR^2(\cos 2\theta + i\sin 2\theta)} Rie^{i\theta} d\theta = \int_0^{\pi/4} e^{iR^2\cos 2\theta - R^2 \sin 2\theta} iRe^{i\theta} d\theta.$$

We now use (a) $\sin\theta \geq (2/\pi)\theta$. This implies that

$$\left|\int_{\gamma_R} e^{iz^2} dz\right| \leq \int_0^{\pi/4} e^{-R^2\sin 2\theta} R d\theta \leq \int_0^{\pi/4} e^{-R^2(4/\pi)\theta} R d\theta = \left[\frac{e^{-R^2(4/\pi)\theta}}{-R^2(4/\pi)} R\right]_0^{\pi/4} = \frac{\pi}{4R}(1 - e^{-R^2}),$$

which clearly tends to 0, as $R \to \infty$.

8. Show that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Hint: Integrate $f(z) = e^{iz}/z$ in the indented semicircular contour.

We set $f(z) = e^{iz}/z$. This function is holomorphic on and inside the contour C. Cauchy's theorem gives

$$\int_C f(z)\,dz = 0.$$

On the horizontal segment $[r, R]$ the contribution to $\int_C f(z)\,dz$ is

$$\int_r^R \frac{e^{ix}}{x}\,dx.$$

On the horizontal segment $[-R,-r]$ the contribution to $\int_C f(z)\,dz$ is

$$\int_{-R}^{-r} \frac{e^{ix}}{x}\,dx = \int_R^r \frac{e^{-iy}}{-y}(-dy) = -\int_r^R \frac{e^{-ix}}{x}dx,$$

using the substitution $y=-x$. The combined contribution by the two segments on the real axis is:

$$\int_r^R \frac{e^{ix}}{x}\,dx - \int_r^R \frac{e^{-ix}}{x}dx = \int_r^R \frac{e^{ix} = e^{-ix}}{x}dx = \int_r^R \frac{2i\sin x}{x}dx \to 2i\int_0^\infty \frac{\sin x}{x}dx,$$

as $r\to 0$ and $R\to\infty$. On the large semicircle γ_R given by $z=Re^{it}$, $t\in[0,2\pi]$, we observe that

$$|e^{iz}| = |e^{iRe^{it}}| = |e^{iR\cos t - R\sin t}| = e^{-R\sin t}.$$

On $[0,\pi/2]$ we use the inequality $\sin t \geq 2t/\pi$ proved in the previous homework in relation to the Fresnel Integrals. We get

$$|e^{iz}| \leq e^{-R2t/\pi}.$$

This contribution is, therefore,

$$\leq \int_0^{\pi/2} \frac{|e^{iz}|}{|z|}R|ie^{it}|dt \leq \int_0^{\pi/2} \frac{e^{-R2t/\pi}}{R}Rdt = \left[\frac{e^{-R2t/\pi}}{-R2/\pi}\right]_0^{\pi/2} = \frac{\pi}{2R} - \frac{\pi}{2R}e^{-R} \to 0$$

as $R\to\infty$. The contribution from $[\pi/2,\pi]$ can be bounded as follows

$$\left|\int_{\pi/2}^{\pi} \frac{e^{iRe^{it}}}{Re^{it}}Rie^{it}dt\right| \leq \int_{\pi/2}^{\pi} \frac{e^{-R\sin t}}{R}Rdt = \int_0^{\pi/2} e^{-R\sin u}du \leq \int_0^{\pi/2} e^{-R2u/\pi}du \to 0,$$

as $R\to\infty$. In the last integral we made the change of variables: $t=\pi-u$ which gives $\sin t = \sin u$.

For the contribution from the small circle γ_r with $z=re^{it}$, $t\in[0,\pi]$, we first write the Taylor series

$$e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots$$

from which it follows that

$$e^{iz} = 1 + E(z)$$

with $E(z)$ a convergent power series on $\mathbb{C}$ with $E(0)=0$. Therefore $E(z)$ is continuous at 0 and, therefore, bounded in a small disc close to 0 by, say M. This means: $\max_{|z|<1}|E(z)| = M$. Therefore,

$$\int_{\gamma_r} \frac{e^{iz}}{z}dz = \int_{\gamma_r}\frac{1}{z}dz + \int_{\gamma_r}E(z)dz = \int_0^{\pi}\frac{1}{re^{it}}rie^{it}dt + \int_{\gamma_r}E(z)dz = i\pi + \int_{\gamma_r}E(z)dz.$$

However,

$$\lim_{r\to 0}\int_{\gamma_r} E(z)dz = 0$$

as

$$\left|\int_{\gamma_r} E(z)dz\right| \leq L(\gamma_r)\max_{z\in\gamma_r}|E(z)| = \pi r\max_{z\in\gamma_r}|E(z)| \leq \pi r M \to 0,$$

as $r \to 0$. We deduce that

$$\lim_{r\to 0}\int_{\gamma_r} \frac{e^{iz}}{z}dz = \pi i.$$

Combining all contributions and letting $r \to 0$ and $R \to \infty$ we get

$$2i\int_0^\infty \frac{\sin x}{x}\,dx = \pi i \Longrightarrow \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Math 2101

Homework 8

Due: December 4, 2013

1. $*$ Explain why the following integrals have the given value using the residue theorem. Complete explanations are required.

$$\text{(a)} \quad \int_0^{2\pi} \frac{1}{3+\sin(t)}\,dt = \frac{\pi}{\sqrt{2}}$$

We consider the function

$$f(z) = \frac{1}{3+(z-z^{-1})/(2i)}\frac{1}{iz}$$

and as contour the unit circle $|z| = 1$ traversed anticlockwise. We rewrite $f(z)$ as

$$f(z) = \frac{1}{3iz + z^2/2 - 1/2} = \frac{2}{6iz + z^2 - 1}$$

from which we see that the poles of the integrand are the roots of $z^2 + 6iz - 1$. We solve the quadratic equation:

$$z^2 + 6iz - 1 = 0 \Leftrightarrow (z+3i)^2 + 8 = 0 \Leftrightarrow z + 3i = \pm i\sqrt{8} \Leftrightarrow z = -3i \pm i\sqrt{8}.$$

From the two roots obviously $i(-3-\sqrt{8})$ is outside the unit circle. The other root $i(-3+\sqrt{8})$ is inside, as

$$-1 < -3 + \sqrt{8} < 0 \Leftrightarrow 2 < \sqrt{8} < 3 \Leftrightarrow 4 < 8 < 9.$$

We have te factorization $z^2 + 6iz - 1 = (z + (3+\sqrt{8})i)(z + i(3-\sqrt{8}))$. We calculate the residue for the simple pole $i(-3+\sqrt{8})$.

$$\begin{aligned}\operatorname{res}(f, i(-3+\sqrt{8})) &= \lim_{z\to i(\sqrt{8}-3)} (z + i(3-\sqrt{8}))f(z) = \lim_{z\to i(\sqrt{8}-3)} \frac{2}{(z + i(3+\sqrt{8}))} \\ &= \frac{2}{(\sqrt{8}-3+3+\sqrt{8})i} = \frac{1}{i\sqrt{8}}.\end{aligned}$$

With the parametrization $z = e^{it}$, $0 \le t \le 2\pi$ we see that $dz = ie^{ie}\,dt$ and

$$\int_{|z|=1} f(z)\,dz = \int_0^{2\pi} \frac{1}{3+(e^{it}-e^{-it})/(2i)}\frac{1}{ie^{it}} ie^{it}\,dt = \int_0^{2\pi} \frac{1}{3+\sin(t)} dt.$$

On the other hand the residue theorem gives

$$\int_{|z|=1} f(z)\,dz = 2\pi i \cdot \operatorname{res}(f, i(\sqrt{8}-3)) = 2\pi i \frac{1}{i\sqrt{8}} = \frac{\pi}{\sqrt{2}}.$$

$$\text{(b)} \quad \int_0^\infty \frac{\log x}{x^2+1}\,dx = 0$$

We take

$$f(z) = \frac{\log z}{1+z^2}$$

with nonprincipal branch of $\log z$ given by

$$\log z = \log|z| + i\arg z, \quad -\pi/2 < \arg z < 3\pi/2.$$

Inside the contour there is a simple pole at i. We compute the residue

$$\operatorname{Res}(f,i) = \lim_{z\to i}(z-i)\frac{\log z}{(z-i)(z+i)} = \lim_{z\to i}\frac{\log z}{(z+i)} = \frac{\log i}{2i} = \frac{\log|i| + i\arg(i)}{2i} = \frac{i\pi/2}{2i} = \frac{\pi}{4}.$$

For the integral over γ_R we have ($|z^2+1| \ge |z^2| - 1 = R^2 - 1$)

$$\left|\int_{\gamma_R} \frac{\log z}{1+z^2}dz\right| \le \pi R\frac{\log R + \pi}{R^2-1} \to 0, \quad R \to \infty.$$

For the integral over γ_r we have ($|z^2+1| \ge 1 - |z^2| = 1 - r^2$)

$$\left|\int_{\gamma_r} \frac{\log z}{1+z^2}dz\right| \le \pi r\frac{\log r + \pi}{(1-r^2)} \to 0, \quad r \to 0.$$

Notice that we used that $r\log r \to 0$, as $r \to 0$, which was proved in the lectures. The residue theorem gives

$$\int_r^R \frac{\log x dx}{1+x^2} + \int_{-R}^{-r} \frac{\log|x| + i\pi}{1+x^2}dx + \int_{\gamma_R}\frac{\log z}{1+z^2}dz + \int_{\gamma_r}\frac{\log z}{1+z^2}dz = 2\pi i \operatorname{Res}(f,i).$$

We take the limit as $R \to \infty$, $r \to 0$ to get

$$\int_0^\infty \frac{\log x dx}{1+x^2} + \int_{-\infty}^0 \frac{\log|x|}{1+x^2}dx + i\pi\int_{-\infty}^0 \frac{dx}{1+x^2} = 2\pi i\frac{\pi}{4}.$$

We substitute $y = -x$ in the second integral, take real parts (notice that $\int_{-\infty}^0 (1+x^2)^{-1}dx$ is real) to get

$$2\int_0^\infty \frac{\log x}{1+x^2}dx = \Re(i\pi^2/2) = 0 \Longrightarrow \int_0^\infty \frac{\log x}{1+x^2}dx = 0.$$

$$\text{(c)} \quad \int_0^{2\pi} \frac{dt}{(a+\cos t)^2} = 2\pi a/(a^2-1)^{3/2} \quad (a>1).$$

We use the contour in Figure 1. The function to integrate is

$$f(z) = \frac{1}{(a+(z+z^{-1})/2)^2}\frac{1}{z} = \frac{4}{(2a+z+z^{-1})^2}\frac{1}{z} = \frac{4z}{(z+z^{-1}+2a)^2z^2} = \frac{4z}{(z^2+2az+1)^2}.$$

We solve $z^2+2az+1=0$ by completing the square

$$(z+a)^2+1-a^2=0 \Longrightarrow z+a=\pm\sqrt{a^2-1} \Longrightarrow z_{1,2}=-a\pm\sqrt{a^2-1}.$$

The two roots are real, as $a>1$ is given. They have negative sum and product 1. This means that only one is inside the circle $|z|=1$. Call it $z_1=-a+\sqrt{a^2-1}$, while $z_2=-a-\sqrt{a^2-1}<-1$. The pole at z_1 is double. We calculate the residue

$$\operatorname{Res}(f,z_1) = \lim_{z\to z_1}\frac{d}{dz}\frac{(z-z_1)^2 4z}{(z-z_1)^2(z-z_2)^2} = \lim_{z\to z_1}\frac{4(z-z_2)-2\cdot 4z}{(z-z_2)^3} = \lim_{z\to z_1}\frac{-4(z+z_2)}{(z-z_2)^3}$$

$$= \frac{-4(z_1+z_2)}{(z_1-z_2)^3} = \frac{-4(-2a)}{(2\sqrt{a^2-1})^3} = \frac{a}{(a^2-1)^{3/2}}.$$

The residue theorem gives

$$\int_{|z|=1} f(z)dz = \int_0^{2\pi}\frac{1}{(a+\cos t)^2}\frac{1}{e^{it}}ie^{it}dt = 2\pi i\operatorname{Res}(f,i) = \frac{2\pi i a}{(a^2-1)^{3/2}}$$

$$\Longrightarrow \int_0^{2\pi}\frac{1}{(a+\cos t)^2}dt = \frac{2\pi a}{(a^2-1)^{3/2}}.$$

2. Explain why the following integrals have the given value using the residue theorem. Complete explanations are required.

$$\text{(a)} \int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^{n+1}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)\pi}{2\cdot 4\cdot 6\cdots(2n)},$$

$$\text{(b)} \int_0^{\pi/2}\frac{1}{a+\cos^2 t}\,dt = \frac{\pi}{2\sqrt{a^2+a}}, \quad a>0.$$

(a) We consider $f(z)=1/(z^2+1)^{n+1}$ and the contour of Figure 2. The only relevant pole is at i but it is of multiplicity $n+1$. We remark that

$$\frac{d^n}{dz^n}(z+i)^{-n-1} = (-n-1)(-n-2)\cdots(-n-n)(z+i)^{-n-1-n} = (n+1)(n+2)\cdots(2n)(-1)^n(z+i)^{-2n-1}.$$

This gives

$$\operatorname{Res}(f,i) = \frac{1}{n!}\frac{d^n}{dz^n}\left((z-i)^{n+1}\frac{1}{(z-i)^{n+1}(z+i)^{n+1}}\right)_{|z=i} = \frac{1}{n!}(n+1)(n+2)\cdots(2n)(-1)^n(2i)^{-2n-1}$$

$$= \frac{(n+1)(n+2)\cdots(2n)}{n!}(-1)^n 2^{-2n-1}(-i)^{2n+1} = \frac{(n+1)(n+2)\cdots(2n)}{n!}(-1)^n \frac{1}{2^{2n+1}}(-i)(-1)^n$$
$$= -i\frac{(n+1)(n+2)\cdots(2n)}{n!2^{2n+1}}.$$

We estimate that the integral over the semicircle γ_R tends to 0 as $R \to \infty$:

$$\left|\int_{\gamma_R} \frac{dz}{(1+z^2)^{n+1}}\right| \leq \pi R \frac{1}{(R^2-1)^{n+1}} \to 0, \quad R \to \infty.$$

The residue theorem gives

$$\int_{-R}^{R} \frac{dx}{(1+x^2)^{n+1}} + \int_{\gamma_R} \frac{dz}{(1+z^2)^{n+1}} = 2\pi i \mathrm{Res}(f, i).$$

Taking the limit as $R \to \infty$ we get

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = 2\pi i(-i)\frac{(n+1)(n+2)\cdots 2n}{n!2^{n+1}} = 2\pi \frac{1\cdot 2\cdots n(n+1)(n+2)\cdots(2n)}{2^{2n}2\cdot 1\cdot 2\cdots n\cdot 1\cdot 2\cdots n}$$

$$= \pi \frac{1\cdot 2\cdots 2n}{2\cdot 4\cdots(2n)\cdot 2\cdot 4\cdots(2n)} = \pi \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}.$$

Alternative method: The substitution $x = \tan\theta$ gives, as $1 + x^2 = (\cos\theta)^{-2}$, $dx = (\cos\theta)^{-2}d\theta$,

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \int_{-\pi/2}^{\pi/2} (\cos^2\theta)^{n+1}\frac{d\theta}{\cos^2\theta} = \int_{-\pi/2}^{\pi/2} (\cos\theta)^{2n} d\theta = \pi\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}$$

by problem 6a in Homework 5. Notice that $\cos^2\theta$ takes the same values on any quadrant.

(b) Since

$$\int_0^{\pi/2} \frac{dt}{a+\cos^2 t} = \frac{1}{4}\int_0^{2\pi} \frac{dt}{a+\cos^2 t},$$

as $\cos^2 t$ takes the same values in every quadrant, we consider the function

$$f(z) = \frac{1}{a + \left(\frac{z+z^{-1}}{2}\right)^2}\frac{1}{iz}$$

on the contour of Figure 2.

We get

$$f(z) = \frac{1}{a + \frac{(z+z^{-1})^2}{4}}\frac{1}{iz} = \frac{1}{a + z^2/4 + z^{-2}/4 + 2/4}\frac{1}{iz} = \frac{-i4z}{(4a+2)z^2 + z^4 + 1}.$$

We solve the biquadratic equation $z^4 + (4a+2)z^2 + 1 = 0$.

$$z^2 = -(2a+1) \pm \sqrt{(2a+1)^2 - 1} = -(2a+1) \pm 2\sqrt{a^2+a}.$$

There are two solutions for z^2 but only is inside the circle, since the solutions are real and negative and their product is 1. Clearly this is the larger solution $b^2 = -(2a+1) + 2\sqrt{a^2+a}$. This gives two solutions for z, $z = \pm b$. The poles are simple and inside $|z| = 1$. We set $c^2 = -(2a+1) - 2\sqrt{a^2+a}$.

$$\operatorname{Res}(f,b) = \lim_{z\to b} \frac{-i(z-b)4z}{-(z^2-b^2)(z^2-c^2)} = \lim_{z\to b} \frac{-i4z}{(z+b)(z^2-c^2)} = \frac{-i4b}{2b(b^2-c^2)} = \frac{-i}{2\sqrt{a^2+a}}$$

$$\operatorname{Res}(f,-b) = \lim_{z\to -b} \frac{-i(z+b)4z}{(z^2-b^2)(z^2-c^2)} = \lim_{z\to -b} \frac{-i4z}{(z-b)(z^2-c^2)} = \frac{4ib}{-2b(b^2-c^2)} = \frac{-i}{2\sqrt{a^2+a}}$$

The residue theorem gives

$$\int_{|z|=1} f(z)\,dz = 2\pi i(\operatorname{res}(f,b) + \operatorname{res}(f,-b)) = 2\pi i 2 \cdot \frac{-i}{2\sqrt{a^2+a}} = \frac{2\pi}{\sqrt{a^2+a}}.$$

We get

$$\int_0^{\pi/2} \frac{dt}{a+\cos^2 t} = \frac{1}{4}\int_0^{2\pi} \frac{dt}{a+\cos^2 t} = \frac{1}{4}\int_{|z|=1} f(z)\,dz = \frac{1}{4}\frac{2\pi}{\sqrt{a^2+a}} = \frac{\pi}{2\sqrt{a^2+a}}.$$

3. Let $p(z)$ and $q(z)$ be holomorphic in the disk $D(z_0, r)$, $r > 0$. Assume that $q(z_0) = 0$, while $q'(z_0) \neq 0$. Assume that $p(z_0) \neq 0$.

(a) Explain why $f(z) = p(z)/q(z)$ has a simple pole at z_0.

Since $q'(z_0) \neq 0$, the function $q(z)$ is not identically zero. Therefore, its zero at z_0 is isolated. Moreover, the zero of $q(z)$ at z_0 is simple, as $q'(z_0) \neq 0$. We can factor it to get

$$q(z) = (z - z_0)g(z)$$

for a holomorphic nonvanishing function $g(z)$ in a neighbourhood of z_0. This makes the quotient $p(z)/q(z)$ holomorphic in the punctured disk $D'(z_0, r)$ and z_0 is an isolated singularity. To see that it is a pole we look at

$$\frac{1}{f(z)} = \frac{q(z)}{p(z)} = (z - z_0)g(z)/p(z).$$

The function $g(z)/p(z)$ is holomorphic and nonzero in a neighbourhood of z_0. Here we have used that $p(z_0) \neq 0$. Therefore, $1/f$ has a simple zero at z_0, and f has a simple pole at z_0.

(b) Show that

$$\operatorname{res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}.$$

$$\text{res}\,(f, z_0) = \lim_{z\to z_0}(z-z_0)\frac{p(z)}{q(z)} = \lim_{z\to z_0} p(z) \lim_{z\to z_0}\frac{z-z_0}{q(z)-q(z_0)} = p(z_0)\frac{1}{\lim_{z\to z_0}\frac{q(z)-q(z_0)}{z-z_0}} = p(z_0)\frac{1}{q'(z_0)}.$$

(c) What happens if, instead of $q'(z_0) \neq 0$, we assume that $q'(z_0) = 0$, while $q''(z_0) \neq 0$? Show that in this case

$$\text{res}\,(f, z_0) = 2\frac{p'(z_0)}{q''(z_0)} - \frac{2}{3}\frac{p(z_0)q'''(z_0)}{[q''(z_0)]^2}.$$

In the case $q'(z_0) = 0$ but $q''(z_0) \neq 0$ we see that q has a zero of order 2 at z_0:

$$\begin{aligned} q(z) &= q(z_0) + q'(z_0)(z-z_0) + \frac{q''(z_0)}{2}(z-z_0)^2 + \frac{q'''(z_0)}{3!}(z-z_0)^3 + \cdots \\ &= (z-z_0)^2\left(\frac{q''(z_0)}{2} + \frac{q'''(z_0)}{3!}(z-z_0)\cdots\right) = (z-z_0)^2 g(z), \end{aligned}$$

with

$$g(z) = \frac{q''(z_0)}{2} + \frac{q'''(z_0)}{3!}(z-z_0)\cdots.$$

The convergent power series $g(z)$ with non-zero constant term represents a holomorphic function in a neighbourhood of z_0 which does not vanish. Then

$$\frac{1}{f(z)} = (z-z_0)^2 g(z)/p(z)$$

The function $g(z)/p(z)$ is holomorphic and nonzero in a neighbourhood of z_0. Here we have used that $p(z_0) \neq 0$. Therefore, $1/f$ has a zero of order two at z_0, and f has a pole of order two at z_0. We calculate the residue. We need to calculate first the derivative

$$\frac{d}{dz}\frac{(z-z_0)^2 p(z)}{q(z)} = \frac{d}{dz}\frac{p(z)}{g(z)} = \frac{p'(z)}{g(z)} + \frac{-p(z)g'(z)}{g(z)^2}$$

We have $g(z_0) = q''(z_0)/2$ and $g'(z_0) = q'''(z_0)/6$. This gives

$$\begin{aligned} \text{res}(f, z_0) &= \lim_{z\to z_0}\frac{d}{dz}\frac{(z-z_0)^2 p(z)}{q(z)} = \lim_{z\to z_0}\frac{p'(z)}{g(z)} + \frac{-p(z)g'(z)}{g(z)^2} = \frac{p'(z_0)}{q''(z_0)/2} - \frac{p(z_0)q'''(z_0)/6}{(q''(z_0)/2)^2} \\ &= 2\frac{p'(z_0)}{q''(z_0)} - \frac{2}{3}\frac{p(z_0)q'''(z_0)}{[q''(z_0)]^2}. \end{aligned}$$

(d) Identify all the isolated singular points of $f(z) = \cot z$. Which ones are poles? Find the corresponding residues.

Since $f(z) = \cos z/\sin z$, we set $p(z) = \cos z$ and $q(z) = \sin z$. We get poles at the points where $\sin z = 0$, and these are the points $z_n = \pi n$, $n \in \mathbb{Z}$. As $\sin'(z) = \cos z$ and $\cos z_n \neq 0$, the poles are simple according to (a) and the residues are according to (b)

$$\text{res}(f, z_n) = \frac{\cos(z_n)}{\cos(z_n)} = 1.$$

4. $*$ Show that there does not exist a holomorphic function f on $D(0,1)$ such that

$$f(1/n) = \begin{cases} 1 + 2/n, & n \quad \text{even}, \\ 1, & n \quad \text{odd}. \end{cases}$$

We notice that $1/n \to 0 \in D(0,1)$. The function $f(z)$ agrees with $g(z) = 1 + 2z$ for $z = z_n = \frac{1}{n}$, n even. By the principle of analytic continuation $f(z) = 1 + 2z$ for all $z \in D(0,1)$. However, this gives for odd n that $f(1/n) = 1 + 2/n \neq 1$. This is a contradiction, so such holomorphic function f does not exist.

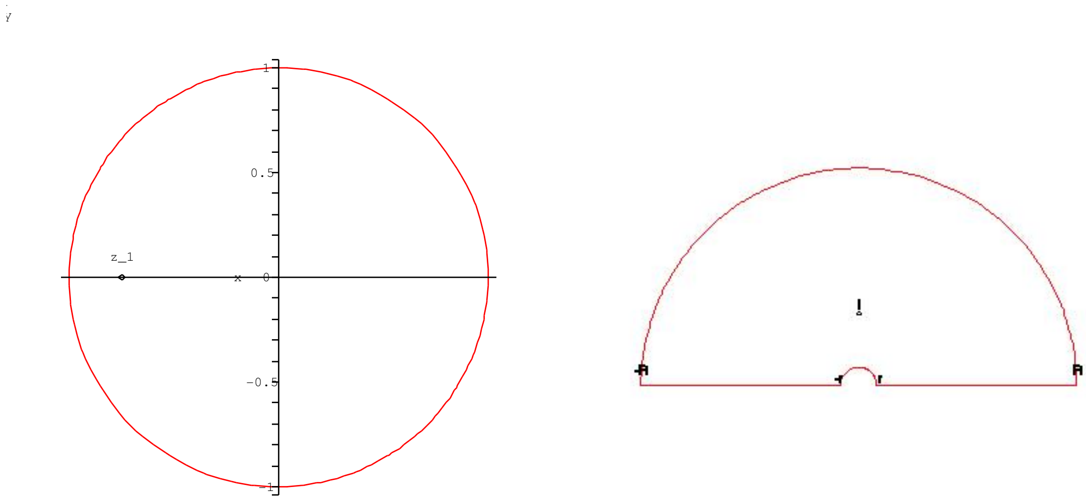


Figure 1: Contour for 1(a) and 1(c), contour for 1(b)

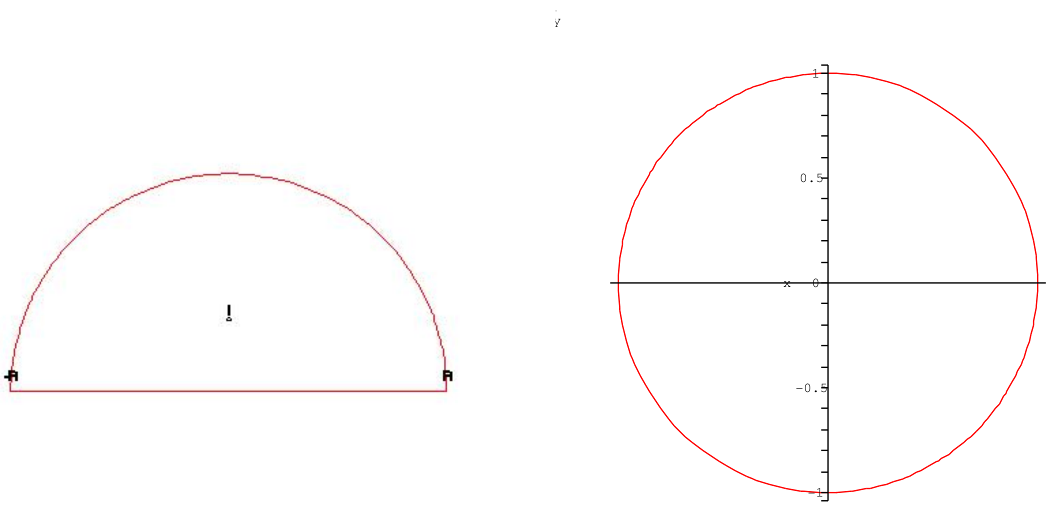


Figure 2: Contours for problem 2(a) and 2(b)

Math 2101

Homework 9

NOT DUE

While this assignment will not be marked, please study Part A for the final exam in May.

PART A

1. (a) How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2?

   Let $f(z) = z^4$ and $g(z) = -6z + 3$. On the circle $|z| = 2$ we have

   $$|f(z)| = |z|^4 = 2^4 = 16, \quad |g(z)| = |-6z + 3| \leq 6|z| + 3 = 6 \cdot 2 + 3 = 15 < |f(z)|.$$

   By Rouché's theorem, f has the same number of zeros inside the circle $|z| = 2$ as $f(z) + g(z) = z^4 - 6z + 3$. Since $f(z)$ has a fourth order zero at 0 and no other zeros, we get for the given polynomial 4 zeros inside $|z| = 2$. We need to subtract the number of zeros inside $|z| = 1$. Now we set $f(z) = -6z$, $g(z) = z^4 + 3$. We get on $|z| = 1$:

   $$|f(z)| = |-6z| = 6|z| = 6, \quad |g(z)| = |z^4 + 3| \leq |z|^4 + 3 = 1 + 3 = 4 < 5 = |f(z)|.$$

   Since $f(z)$ has one simple zero at 0, we get that $f(z) + g(z) = -6z + z^4 + 3$ has one zero inside $|z| = 1$. As a result there are 3 zeros inside the annulus.

   (b) Find the number of the roots of the equation

   $$z^6 - 5z^4 + 8z - 1 = 0$$

   in the annulus $\{z : 1 < |z| < 2\}$.

   For the disc $|z| \leq 1$ we use $f(z) = 8z$, $g(z) = z^6 - 5z^4 - 1$, so that on $|z| = 1$ we have

   $$|f(z)| = |8z| = 8, \quad |g(z)| = |z^6 - 5z^4 - 1| \leq |z|^6 + 5|z|^4 + 1 = 1 + 5 + 1 = 7 < 8 = |f(z)|.$$

   Since $f(z)$ has one zero at 0, the function $f(z) + g(z) = z^6 - 5z^4 + 8z - 1$ has the same number of zeros inside $|z| = 1$, i.e. 1. (Rouché's theorem).

   For the disc $|z| = 2$, the above technique does not work. With any choice of splitting in one plus three or two plus two terms of the polynomial, we cannot apply Rouché's theorem. Here are the calculations:

   $$|z^6| = 2^6 = 64, \quad |-5z^4 + 8z - 1| \leq 5 \cdot 2^4 + 8 \cdot 2 + 1 = 80 + 16 + 1 > 64,$$

$$|-5z^4| = 5\cdot 16 = 80, \quad |z^6+8z-1| \le 2^6 + 8\cdot 2 + 1 = 64+16+1 = 81 > 80,$$
$$|8z| = 16, \quad |z^6 - 5z^4 - 1| \le 2^6 + 5\cdot 2^4 + 1 = 64 + 80 + 1 > 16,$$
$$|1| = 1, \quad |z^6 - 5z^4 + 8z| \le 2^6 + 5\cdot 16 + 16 > 1,$$
$$|z^6 - 5z^4| = |z|^4|z^2-5| \ge 16\cdot 1 = 16, \quad |8z-1| \le 8|z| + 1 = 17 > 16,$$
$$|z^6 + 8z| = |z||z^5+8| \ge 2\cdot(32-8) = 48, \quad |-5z^4-1| \le 80+1 > 48,$$
$$|z^6 - 1| \ge 63, \quad |-5z^5 + 8z| = |z||-5z^4+8| \le 2\cdot(80+8) > 63.$$

Here is a trick. We choose a smaller disc of radius $3/2$ and find the number of zeros inside $|z| = 3/2$. We set $f(z) = -5z^4$, $g(z) = z^6 + 8z - 1$, so that on $|z| = 3/2$ we have

$$|f(z)| = 5(3/2)^4 = 405/16, \quad |g(z)| \le (3/2)^6 + 8\cdot 3/2 + 1 = (3/2)^6 + 13 < 405/16,$$

as

$$(3/2)^6 < 197/16 \Leftrightarrow (81/16)(9/4) < 197/16 \Leftrightarrow 729/4 < 197 \Leftrightarrow 729 < 788.$$

By Rouché's theorem $5z^4$ has the same number of zeros as $z^6 - 5z^4 + 8z - 1$ inside $|z| = 3/2$, i.e. 4. However, this does not suffice to guarantee that inside the given annulus $1 < |z| < 2$ there are 3 zeros. It is possible that we have one or two more zeros in the annulus $3/2 < |z| < 2$. We use the intermediate value theorem on the real axis:

$$f(-3) = 299, \quad f(-2) = -33, \quad f(2) = -1, \quad f(3) = 347.$$

By the intermediate value theorem there exist one solution in $(-3,-2)$ and another in $(2,3)$. This accounts for all 6 roots. With a computer program one can compute the roots to be

$$0.1251528553, 1.296061165, 2.023328803, -0.5358533086 + 0.9984804513i,$$

$$-2.372836206, -0.5358533086 - 0.9984804513i.$$

2. Let C be the unit circle $|z| = 1$ traversed anticlockwise. Determine the variation of the argument $\Delta_C \arg f(z)$ for the functions

$$\text{(a) } f(z) = z^2, \quad \text{(b) } f(z) = \frac{z^3+2}{z}.$$

According to the formula from the lectures

$$\Delta_C \arg f(z) = \frac{1}{i}\int_C \frac{f'(z)}{f(z)}\,dz.$$

For (a) we have

$$\Delta_C \arg f(z) = \frac{1}{i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{i}\int_C \frac{2z}{z^2}\,dz = \frac{2}{i}\int_C \frac{1}{z}\,dz = \frac{2}{i}2\pi i = 4\pi.$$

For (b) we have observe that

$$\log f(z) = \log(z^3+2) - \log z \Longrightarrow \frac{f'(z)}{f(z)} = \frac{3z^2}{z^3+2} - \frac{1}{z}.$$

Therefore,

$$\Delta_C \arg f(z) = \frac{1}{i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{i}\int_C \left(\frac{3z^2}{z^3+2} - \frac{1}{z}\right)\,dz = \frac{1}{i}\int_C \frac{3z^2}{z^3+2}\,dz - \frac{1}{i}\int_C \frac{1}{z}\,dz$$
$$= 0 - \frac{1}{i}2\pi i = -2\pi,$$

since the first integral is 0 by Cauchy's theorem. The rational function $3z^2/(z^3+2)$ is holomorphic in a region containing the unit circle and its interior, as the only problems with it are the roots of the denominator. These are the third roots of -2 which have modulus $\sqrt[3]{2} > 1$ so are outside the unit circle.

3. Let λ be real and $\lambda > 1$, Show that the equation

$$ze^{\lambda - z} = 1$$

has exactly one solution in the disc $|z| = 1$, which is real and positive.

We rewrite the equation as

$$ze^{\lambda} = e^z \Leftrightarrow ze^{\lambda} - e^z = 0,$$

so that we choose $f(z) = ze^{\lambda}$ and $g(z) = -e^z$ and apply Rouché's theorem on $|z| = 1$.

$$|f(z)| = |z|e^{\lambda} = e^{\lambda} > e, \quad \text{as } \lambda > 1,$$
$$|g(z)| = |-e^z| = e^{\Re z} \leq e^1 < |f(z)|,$$

so the function $f(z) + g(z) = ze^{\lambda} - e^z$ has the same number of zeros inside $|z| = 1$ as $f(z)$, i.e. 1 solution. To prove that it is real and positive we apply the intermediate value theorem on $[0, 1]$:

$$f(0) + g(0) = 0 - e^0 = -1, \quad f(1) + g(1) = e^{\lambda} - e^1 > 0$$

as $\lambda > 1$. So there exists a solution in $[0, 1]$.

4. Suppose that $f(z)$ is holomorphic in a punctured disc $D'(z_0, \delta)$. Suppose that for some constant M and for all $z \in D'(z_0, \delta)$ we have

$$|f(z)| \leq M|z - z_0|^{-1/4}.$$

Show that the singularity of f at z_0 is removable.

Consider $g(z) = (z - z_0)f(z)$ which still has an isolated singularity at z_0 and satisfies

$$|g(z)| \leq M|z - z_0|^{3/4}.$$

We see that $g(z)$ is bounded in the punctured disc. By Riemann's theorem, the singularity of g is removable and we can define $g(z_0)$ so that g is holomorphic in the whole disc. Moreover, the inequality above gives $\lim_{z \to z_0} g(z) = 0$, so that $g(0) = 0$ is the only choice, i.e. g has a zero at z_0. We write $g(z) = (z - z_0)^N h(z)$ for N the order of the zero and h holomorphic and nonzero in a small disc $D(z_0, \epsilon)$ around z_0. Then

$$f(z) = (z - z_0)^{-1}(z - z_0)^N h(z) = (z - z_0)^{N-1} h(z).$$

Since h is bounded on $D(z_0, \epsilon)$ and $N \geq 1$ we see that f is bounded on $D'(z_0, \epsilon)$. By Riemann's theorem its singularity is removable.

PART B

This part contains a few harder problems that are interesting and aimed for advanced study.

1. (Schwarz's Lemma) Let f be holomorphic on the unit disc $\{z, |z| < 1\}$ with

   (a) $|f(z)| \leq 1$ for $|z| < 1$,

   (b) $f(0) = 0$.

   Show that $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for $|z| < 1$. Moreover, if $|f'(0)| = 1$ or $|f(w)| = |w|$ for some point w with $0 < |w| < 1$, then we can find a c with $|c| = 1$ and

   $$f(z) = cz, \quad \forall z \in \mathbb{C}, |z| < 1.$$

   Let $g(z) = f(z)/z$ for $z \neq 0$. Then g is holomorphic in the punctured unit disc. As $f(0) = 0$, we see by the definition of the derivative $f'(0)$ that $\lim_{z \to 0} g(z) = 0$. Therefore, g is bounded in the punctured unit disc. By Riemann's theorem the singularity at 0 is removable and, if we set $g(0) = f'(0)$, then g is holomorphic on the whole disc. Moreover, for any r with $0 < r < 1$ and $|z| = r$ we have

   $$|g(z)| = \frac{f(z)|}{|z|} \leq \frac{1}{r},$$

By the maximum modulus theorem, $|g(z)| \leq 1/r$ for all $|z| \leq r$. We let $r \to 1$ to deduce that $|g(z)| \leq 1$ on the unit disc. This implies

$$|g(0)| = |f'(0)| \leq 1, \quad \frac{|f(z)|}{|z|} = |g(z)| \leq 1 \Longrightarrow |f(z)| \leq |z|.$$

If $|f'(0)| = 1$ or $|f(w)| = |w|$ for a nonzero w in the unit disc, we get that $|g(0)| = 1$ or $|g(w)| = 1$ at an interior point of the unit disc. The maximum modulus principle implies that g is constant, say $g(z) = c$. Moreover we see that $|c| = 1$. Then $f(z) = cz$.

Remark: This lemma is very important and leads to the fact that linear fractional transformations, while they do not preserve ordinary distance between points, they preserve a different kind of distance coming out of the hyperbolic metric. The lemma forms the basis for the connection of complex analysis with hyperbolic geometry.

2. Prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{\sin^2(\pi u)} \quad (u \notin \mathbb{Z}), \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

using the function $f(z) = \dfrac{\pi \cot(\pi z)}{(u+z)^2}$ integrated over the boundary of the square $[-(N+1/2), N+1/2] \times [-(N+1/2), N+1/2]$, $N \geq |u|$, $N \in \mathbb{N}$. This is one of the many derivations of the value $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, due originally to Euler.

Remark: This problem and the technique introduced using $\pi \cot(\pi z)$ allows to find the sum of lots of series that cannot be easily found using real analysis. We can e.g sum series of the form $\sum f(n)$ for $f(z)$ having poles which are not integers, and is a rational function satisfying $|f(z)| \leq M|z|^{-2}$ for $|z|$ large.

The function

$$g(z) = \pi \cot(\pi z) = \frac{\pi \cos(\pi z)}{\sin(\pi z)}$$

has poles at $n \in \mathbb{Z}$. Since $(\sin(\pi z))' = \pi \cos(\pi z)$, which does not vanish at the integers, the poles are simple. We calculate the residues

$$\operatorname{Res}(g, n) = \lim_{z \to n} (z-n) \frac{\pi \cos(\pi z)}{\sin(\pi z)} = \lim_{z \to n} \frac{(z-n)\pi \cos(\pi z)}{\sin(\pi z) - \sin(\pi n)} = \frac{\pi \cos \pi n}{\pi \cos \pi n} = 1,$$

using the definition of the derivative of $\sin(\pi z)$ at n. If $u \notin \mathbb{Z}$, the function

$$f(z) = g(z) \frac{1}{(z+u)^2}$$

has poles at the integers and has an extra pole at $-u$. This is a double pole with residue

$$\operatorname{Res}(f, -u) = \lim_{z \to -u} \frac{d}{dz} (z+u)^2 f(z) = \lim_{z \to -u} g'(z) = \lim_{z \to -u} \frac{-\pi^2}{\sin^2(\pi z)} = -\frac{\pi^2}{\sin^2(\pi u)}.$$

Moreover,

$$\operatorname{Res}(f,n) = \frac{1}{(n+u)^2}, \qquad n \in \mathbb{Z}.$$

This formula is true for $n \neq 0$ and $u = 0$ as well.

There is a modification needed for $u = 0$: In this case $f(z)$ has a triple pole at 0. We have

$$\operatorname{Res}(f,0) = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2} z^3 \frac{\pi\cot(\pi z)}{z^2} = \frac{\pi}{2}\lim_{z\to 0}\frac{z\cos\pi z}{\pi z - (\pi z)^3/6 + \cdots} = \frac{1}{2}\lim_{z\to 0}\frac{\cos\pi z}{1 - \pi^2 z^2/6 + \cdots}$$

using the Taylor series of $\sin\pi z$. The easiest way to proceed is to find the reciprocal the power series in the denominator:

$$\left(1 - \frac{\pi^2 z^2}{6} + \frac{\pi^4 z^4}{5!} - \cdots\right)^{-1} = 1 + \frac{\pi^2}{6} z^2 + \cdots$$

The technique is to assume that the reciprocal has a power series $\sum a_n z^n$ and match coefficients.

$$\left(1 - \frac{\pi^2 z^2}{6} + \frac{\pi^4 z^4}{5!} - \cdots\right)(a_0 + a_1 z + a_2 z^2 + \cdots) = 1$$

$$\implies 1 \cdot a_0 = 1, \quad 1 \cdot a_1 + 0 \cdot a_0 = 0, \quad 1 \cdot a_2 - \frac{\pi^2}{6} a_0 = 0, \ldots$$

Now we multiply with the Taylor series of $\cos(\pi z)$ and differentiate twice to get

$$\operatorname{Res}(f,0) = \lim_{z\to 0}\frac{1}{2}\frac{d^2}{dz^2}\left(1 - \frac{\pi^2 z^2}{2} + \cdots\right)\left(1 + \frac{\pi^2 z^2}{6} + \cdots\right)$$

$$= \lim_{z\to 0}\frac{1}{2}\frac{d^2}{dz^2}\left(1 + \pi^2(-1/2 + 1/6)z^2 + \cdots\right) = \frac{1}{2}\cdot 2\pi^2(1/6 - 1/2) = -\frac{\pi^2}{3}.$$

The alternative would be successive differentiation of $z^3 f(z)$, which is not easier.

If we can show that the integrals over the boundary ∂S_{R_N} of the square S_R tend to 0, then the residue theorem gives ($R > |u|$):

$$\int_{\partial S_R} f(z)dz = 2\pi i\left(\sum_{|n|<R}\frac{1}{(n+u)^2} - \frac{\pi^2}{\sin^2(\pi u)}\right)$$

$$\implies 0 = \sum_{n\in\mathbb{Z}}\frac{1}{(n+u)^2} - \frac{\pi^2}{\sin^2(\pi u)},$$

which gives the result for $u \notin \mathbb{Z}$. If $u = 0$, the residue theorem gives:

$$\int_{\partial S_R} f(z)dz = 2\pi i\left(\sum_{|n|<R, n\neq 0}\frac{1}{n^2} - \frac{\pi^2}{3}\right)$$

and finally

$$2\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{3}\Longrightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

On the line $z = N + 1/2 + iy$ we have

$$\left|\frac{\cos\pi z}{\sin\pi z}\right| = \left|\frac{\cos(N+1/2+iy)\pi}{\sin(N+1/2+iy)\pi}\right| = \left|\frac{\cos(N+1/2)\pi\cos(\pi iy)-\sin(N+1/2)\pi\sin(\pi iy)}{\sin(N+1/2)\pi\cos(\pi iy)+\sin(\pi iy)\cos(N+1/2)\pi}\right|$$

$$= \left|\frac{\sin\pi iy}{\cos\pi iy}\right| = \frac{|e^{-\pi y}-e^{\pi y}|}{e^{-\pi y}+e^{\pi y}}$$

which is bounded. On the line $z = x \pm i(N + 1/2)$ we have

$$\left|\frac{\cos\pi z}{\sin\pi z}\right|^2 = \left|\frac{\cos(x\pm i(N+1/2))\pi}{\sin(x\pm i(N+1/2))\pi}\right|^2 = \left|\frac{\cos x\pi\cos(\pi i(N+1/2))\mp\sin x\pi\sin(\pi i(N+1/2))}{\sin x\pi\cos(\pi i(N+1/2))\pm\sin(\pi i(N+1/2))\cos x\pi}\right|^2$$

Since $\sin\pi i(N + 1/2)$ is purely imaginary and

$$\cos\pi i(N+1/2) = \frac{e^{-\pi(N+1/2)}+e^{\pi(N+1/2)}}{2} = \cosh(\pi(N+1/2))$$

$$\sin\pi i(N+1/2) = \frac{e^{-\pi(N+1/2)}-e^{\pi(N+1/2)}}{2i} = \frac{-1}{i}\sinh(\pi(N+1/2))$$

we get

$$\left|\frac{\cos\pi z}{\sin\pi z}\right|^2 = \frac{\cos^2(x\pi)\cosh^2(\pi(N+1/2))+\sin^2(\pi x)\sinh^2(\pi(N+1/2))}{\sin^2(\pi x)\cosh^2(\pi(N+1/2))+\cos^2(x\pi)\sinh^2(\pi(N+1/2))}$$

and this expression is bounded for $-(N+1) \leq x \leq N+1/2$ and $N \to \infty$. Let K be an upper bound of $|\cot(\pi z)|$ on the sides of all the squares $S_R = S_{R_N}$. The length of the boundary ∂S_{R_N} of the square is $4 \cdot 2(N + 1/2)$. This gives

$$\left|\int_{\partial S_{R_N}} f(z)dz\right| \leq 8(N+1/2)\pi\frac{K}{(N+1/2-|u|)^2} \to 0, \quad N\to\infty,$$

since on ∂S_{R_N} we have $|z+u| \geq |z| - |u| \geq (N + 1/2) - |u|$ (the closest points to the origin on the sides of the square are the points on the real and imaginary axes).

3. In this problem $\int_{c-i\infty}^{c+i\infty}$ denotes a contour integral along the vertical line $\Re(s) = c$ traversed upwards.

   (a) Prove that for $c > 0$ we have $\displaystyle\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s^2}ds = \begin{cases}\log x, & x>1,\\ 0, & 0<x\leq 1.\end{cases}$

For $x > 1$ we use the contour with the semicircle γ_R on the left in Figure **??**. If R is sufficiently large, the contour contains the double pole at $s = 0$. We compute the residue:

$$\mathrm{Res}(f, 0) = \lim_{s\to 0} \frac{d}{ds}\left(s^2 \frac{x^s}{s^2}\right) = \lim_{s\to 0} x^s \log x = x^0 \log x = \log x.$$

On γ_R we have

$$\left|\int_{\gamma_R} \frac{x^s}{s^2} ds\right| \leq \pi R \frac{1}{(R-c)^2} \to 0, \quad R \to \infty,$$

since on $\gamma_R(t) = c + Re^{it}$, $\pi/2 \leq t \leq 3\pi/2$, $|s^2| = |c + Re^{it}|^2 \geq (R-c)^2$ and $|x^s| = |e^{s\log x}| = e^{\log x \Re(s)} = e^{\log x (c + R\cos t)} \leq x^c$ since $\cos t \leq 0$ on the left semicircle, while $\log x \geq 0$. The choice of this contour depends exactly on this fact that. We apply the residue theorem gives

$$\int_{\gamma_R} \frac{x^s}{s^2} ds + \int_{c-iR}^{c+iR} \frac{x^s}{s^2} ds = 2\pi i \mathrm{Res}(f, 0) = 2\pi i \log x \Longrightarrow \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s^2} ds = 2\pi i \log x,$$

by letting $R \to \infty$.

For $0 < x \leq 1$ we use the contour with the semicircle γ_R on the right in Figure **??**. On γ_R we have

$$\left|\int_{\gamma_R} \frac{x^s}{s^2} ds\right| \leq \pi R \frac{1}{(R-c)^2} \to 0, \quad R \to \infty,$$

since on $\gamma_R(t) = c + Re^{it}$, $-\pi/2 \leq t \leq \pi/2$, $|s^2| = |c + Re^{it}|^2 \geq (R-c)^2$ and $|x^s| = |e^{s\log x}| = e^{\log x \Re(s)} = e^{\log x (c + R\cos t)} \leq x^c$ since $\cos t \geq 0$ on the right semicircle, while $\log x \leq 0$. The choice of this contour depends exactly on this fact that.

Inside the contour there is no pole, so Cauchy's theorem gives

$$\int_{\gamma_R} \frac{x^s}{s^2} ds + \int_{c-iR}^{c+iR} \frac{x^s}{s^2} ds = 0 \Longrightarrow \int_{c-i\infty}^{c+\infty} \frac{x^s}{s^2} ds = 0,$$

by taking $R \to \infty$.

(b) Prove that, for $c > 0$, $\displaystyle\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} ds = \begin{cases} 1, & x > 1, \\ 1/2, & x = 1, \\ 0, & 0 < x < 1. \end{cases}$ (Perron formula)

We use the same contours as in (a) for the same x with the exception of $x = 1$. However, the argument is a bit trickier because we have one power less in the denominator.

For $x > 1$, inside the contour there is a pole at $s = 0$, which is simple:

$$\mathrm{Res}(f, 0) = \lim_{s\to 0} s \frac{x^s}{s} = x^0 = 1.$$

We need to control the integral on γ_R. As above, on the left semicircle $s(t) = c + Re^{it}$, $\pi/2 \le t \le 3\pi/2$.

$$|x^s| = |e^{s \log x}| = e^{\log x(c + R\cos t)}$$

We also remark that the inequality $\sin y \ge 2y/\pi$ holds for $0 \le y \le \pi/2$. See homework 5, 5(a). Now

$$\left|\int_{\gamma_R} \frac{x^s}{s} ds\right| = \left|\int_{\pi/2}^{3\pi/2} \frac{x^c x^{R\cos t}}{c + Re^{it}} iRe^{it} dt\right| = \left|\int_0^{\pi} \frac{x^c x^{-R\sin y}}{c + Re^{i(y+\pi/2)}} Re^{iy} dy\right| \le \int_0^{\pi} \frac{x^c x^{-R\sin y}}{R - c} R dy$$

with the substitution $t = y + \pi/2$. The last integral can be split into two equal integrals over $[0, \pi/2]$ and $[\pi/2, \pi]$, since $\sin y$ takes the same values in both. We get

$$\left|\int_{\gamma_R} \frac{x^s}{s} ds\right| \le 2\int_0^{\pi/2} \frac{Rx^c x^{-R\sin y}}{R - c} dy \le 2\int_0^{\pi/2} \frac{Rx^c x^{-R2y/\pi}}{R - c} dy = \frac{Rx^c}{R - c}\left[\frac{x^{-2Ry/\pi}}{-R(\log x)2/\pi}\right]_0^{\pi/2}$$

$$= \frac{\pi x^c}{(R - c)(\log x)2}\left(-x^{-R} + 1\right) \to 0, \quad R \to \infty,$$

as $x > 1$.

For $0 < x < 1$ the parametrization of the right semicircle is $s(t) = c + Re^{it}$, $-\pi/2 \le t \le \pi/2$. We substitute $t = y - \pi/2$:

$$\left|\int_{\gamma_R} \frac{x^s}{s} ds\right| = \left|\int_{-\pi/2}^{\pi/2} \frac{x^c x^{R\cos t}}{c + Re^{it}} iRe^{it} dt\right| \le \int_{-\pi/2}^{\pi/2} R\frac{x^c x^{R\cos t}}{R - c} dt = 2\int_0^{\pi/2} \frac{Rx^c x^{R\sin y}}{R - c} dy$$

Now $x < 1$, so $\log x < 0$ and

$$x^{R\sin y} = e^{\log x R\sin y} \le e^{\log x R2y/\pi} = x^{2Ry/\pi}.$$

$$\left|\int_{\gamma_R} \frac{x^s}{s} ds\right| \le 2\int_0^{\pi/2} \frac{Rx^c x^{R2y/\pi}}{R - c} dy = \frac{Rx^c}{R - c}\left[\frac{x^{2Ry/\pi}}{R(\log x)2/\pi}\right]_0^{\pi/2}$$

$$= \frac{\pi x^c}{(R - c)(\log x)2}\left(x^R - 1\right) \to 0, \quad R \to \infty,$$

as $x < 1$.

For $x = 1$ we compute the integral directly:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s} ds = \frac{1}{2\pi i}\int_{\infty}^{\infty} \frac{idt}{c + it} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{c}{c^2 + t^2} - \frac{it}{c^2 + t^2} dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{c}{c^2 + t^2} dt$$

as the function $t/(c^2 + t^2)$ is odd. This integral is elementary: substitute $t = cu$ to get

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s} ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{c \cdot cdt}{c^2 + c^2u^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{du}{1 + u^2} = \left[\frac{\arctan(u)}{2\pi}\right]_{-\infty}^{\infty} = \frac{1}{2\pi}\pi = \frac{1}{2}.$$

(c) Let the function $f(s)$ be defined by the absolutely convergent series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \Re(s) > a \geq 0.$$

Show that for $x \notin \mathbb{Z}$

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) \frac{x^s}{s} ds, \quad c > a.$$

Since $|n^s| = n^{\Re(s)}$ and the series converges absolutely for $\Re(s) > a$, we have that the series

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\Re(s)}} < \infty, \quad \Re(s) > a.$$

This gives a stronger notion of convergence, which you will see in Analysis 4. It is called uniform convergence. If we fix $\Re(s) = c > a$, then the convergence of the series in the s variable is uniform (by the Weierstraß test). Moreover, on the vertical line $\Re(s) = c$ we have

$$\left| \frac{x^s}{s} \right| = \frac{x^c}{|s|},$$

which is bounded so we get uniform convergence of the series even when multiplied by x^s/s. The importance of uniform convergence is that it allows to interchange summation and integration to get

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) \frac{x^s}{s} ds = \sum_{n=1}^{\infty} a_n \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s} ds = \sum_{n=1}^{\infty} a_n \left\{ \begin{array}{ll} 0, & x < n, \\ 1, & x > n \end{array} \right\} = \sum_{n<x} a_n.$$

Remark: With some small additional work one can avoid mentioning uniform convergence.

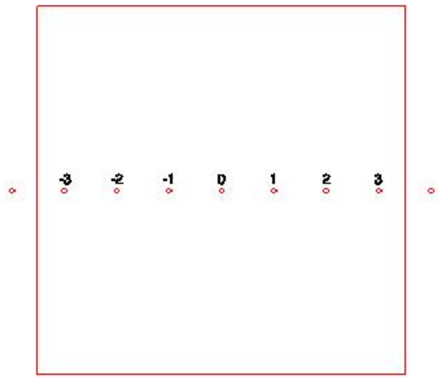


Figure 1: Contour for Problem 2

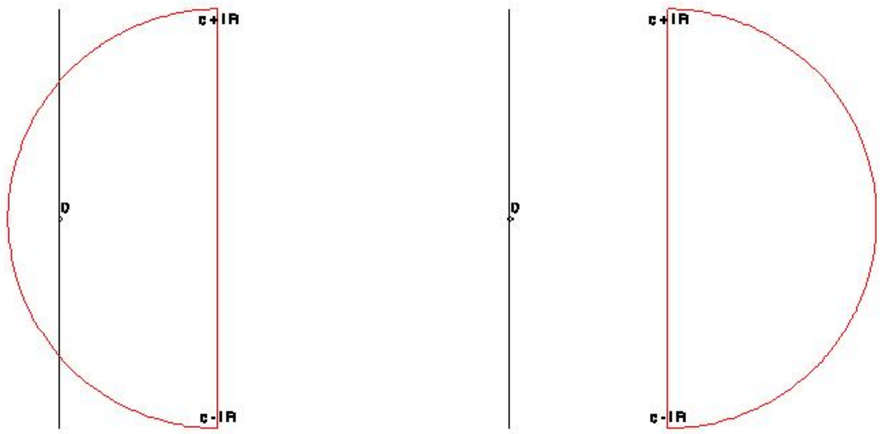


Figure 2: Contours for Problem 3: on the left $x > 1$, on the right $x < 1$.